Shrinkage Estimation Of P(Y<X) In The Weibull Model

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Abstract

We consider the problem of estimating R=P(Y<X) where X and Y have independent Weibull distributions with shape parameter β, but with different scale parameters θ_1 and θ_2 respectively. Assuming that there is a prior guess or estimate R_0, we develop various shrinkage estimators of R that incorporate this prior information. The performance of the new estimators is investigated and compared with the maximum likelihood estimator using Monte Carlo methods. It is found that some of these estimators are very successful in taking advantage of the prior estimate available. Recommendations concerning the use of these estimators are presented.

1. Introduction

The problem of making inference about R=P(Y<X) has received a considerable attention in literature. This problem arises naturally in the context of mechanical reliability of a system with strength X and stress Y. The system fails any time its strength is exceeded by the stress applied to it.

Another interpretation of R is that it measures the effect of the treatment when X is the response for a control group and Y is for the treatment group. Various versions of this problem have been discussed in literature: Enis and Geisser(1971) discussed Bayesian estimation of R when X and Y are exponential. Awad et al. (1981), proposed three estimators of R when X and Y have a bivariate exponential distribution. Tong (1974) derived the MVUE of R where X and Y are exponential. Johnson (1975) gave a correction to the results in Tong (1974). Some other aspects of inference about R are given in AL-Hussaini et al. (1997). In some applications, an experimenter often possesses some knowledge of the experimental conditions based on the behaviour of the system under consideration, or from past experience or some extraneous source, and is thus in position to give an educated guess or an initial estimate of the parameter of interest. Given a prior estimate R_0 of R, we are looking for an estimator that incorporates this information. Those estimators are then called ''shrinkage estimators'' as introduced by Thompson (1968). Balkizi and Dayyeh (2003) discussed different shrinkage estimators of R when X and Y are exponential.

In this article, we shall propose some shrinkage estimators for R when X and Y follows Weibull distribution, in Sec. 2. A Monte Carlo study to investigate the behaviour of these estimators is described in Sec. 3. Results and conclusions are given in the final section.

2. Shrinkage Estimation Procedures

In this study, X and Y have independent Weibull distributions with shape parameter β, but with different scale parameters θ_1 and θ_2 respectively, that is

$f_X(x,\theta_1)=\frac{\beta}{\theta_1}x^{(\beta-1)}\exp(-\frac{x^\beta}{\theta_1})$, x>0;

$$f_X(y,\theta_2)=\frac{\beta}{\theta_2}y^{(\beta-1)}\exp\left(-\frac{y^\beta}{\theta_2}\right), y>0.$$

Here we assumed the shape parameter to be known. Let $X1,\ldots,Xn_1$ be a random sample for X and $Y1,\ldots,Yn_2$ be a random sample for Y. The parameter R we want to estimate is $R = P[Y < X] = \frac{\theta_1}{\theta_1+\theta_2}$. The maximum likelihood estimator of R can be shown to be $\hat{R}=\frac{\hat{\theta}_1}{\hat{\theta}_1+\hat{\theta}_2}$, where $\hat{\theta}_1=\frac{\sum_{i=1}^{n1}x_i^\beta}{n_1}$ and $\hat{\theta}_2=\frac{\sum_{j=1}^{n2}y_j^\beta}{n_2}$. Now we will develop several shrinkage estimators of R that incorporates the experimenters of guess which is R_0. The suggested estimators are of the form $\tilde{R}=c\hat{R}+(1-c)R_0$, $0\le c\le 1$. We will determine the value of c in the following ways;

2.1. Shrinkage towards a Pre-specified R

Here we are looking for c_1 in the estimator $\tilde{R}=c_1\hat{R}+(1-c_1)R_0$ that minimizes its mean square error $MSE(\tilde{R}_1)=E(\tilde{R}_1-R)^2=E[(c_1\hat{R}+(1-c_1)R_0)-R]^2$. The value of c_1 that minimizes this MSE can be shown to be $c_1 =$

$[(R-R_0)(E(\hat{R})-R_0)]/[E(\hat{R}^2)-2R_0E(\hat{R})+R_0^2]$, subject to $0 \le c_1 \le 1$. However this value of c_1 depends on the unknown parameter R. Substituting $\hat{R}$ instead of R we get $\hat{c}_1 = [(\hat{R}-R_0)(E(\hat{R})-R_0)]/[E(\hat{R}^2)-2R_0E(\hat{R})+R_0^2]$. Hence, our shrinkage estimator is $\tilde{R}_1 = \hat{c}_1\hat{R}_1 + (1-\hat{c}_1)R_0$.

We now obtain approximate values of $E(\hat{R})$ and var($\hat{R}$). Notice that $\hat{R} = \frac{\hat{\theta}_1}{(\hat{\theta}_1+\hat{\theta}_2)} = \frac{1}{1+(\hat{\theta}_2/\hat{\theta}_1)}$, and hence $(\hat{\theta}_2/\hat{\theta}_1) = (1/\hat{R}) - 1$. Thus $(\theta_1/\theta_2)(\hat{\theta}_2/\hat{\theta}_1) = (\theta_1/\theta_2)[(1/\hat{R})-1]$. It is shown in the next section that $V = (\theta_1/\theta_2)(\hat{\theta}_2/\hat{\theta}_1) \sim F_{2n_2,2n_1}$. Following Lindley (1969), Balkizi (2003),we get $E(\hat{R}) = (1+(\theta_2/\theta_1)E(V))^{-1} + var(V)(\theta_2/\theta_1)^2(1+$ *θ2θ1EV)−3*, *varR= varVθ2θ12(1+θ2θ1EV)−2* where $E(V) = n_1/(n_1-1)$, $var(V) = [n_1^2(n_1+n_2-1)]/[n_2(n_1-1)(n_1-2)]$; in these formulas θ_1 and θ_2 are further replaced by $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively, for numerical computation.

2.2. Shrinkage Using the p-value of the LRT

For testing $H_0: R = R_0$ vs. $H_1: R \ne R_0$, the likelihood ratio test is the form: reject H_0 when $(\hat{\theta}_2/\hat{\theta}_1) < \alpha_1$ or $(\hat{\theta}_2/\hat{\theta}_1) > \alpha_2$. his follows by noticing that $H_0: R = R_0$ vs. $H_1: R \ne R_0$ is equivalent to $H_0: \theta_1 = R_0\theta_2/(1-R_0)$ vs. $H_1: \theta_1 \ne R_0\theta_2/(1-R_0)$. The MLEs of θ_1 and θ_2 are $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively, while the restricted MLEs of θ_1 and θ_2 are given by $(1/(n_1+n_2))(n_1\hat{\theta}_1 + (R_0/(1-R_0))n_2\hat{\theta}_2)$ and $(1/(n_1+n_2))((R_0/(1-R_0))n_1\hat{\theta}_1 + n_2\theta_2)$, respectively. Application of the likelihood criterion leads directly to the result. Notice that $(2n_1\hat{\theta}_1/\theta_1) \sim > \chi^2_{(2n_1)}$ and $(2n_2\hat{\theta}_2/\theta_2) \sim > \chi^2_{(2n_2)}$; therefore $[(2n_2\hat{\theta}_2/\theta_2)/2n_2]/[(2n_1\hat{\theta}_1/\theta_1)/2n_1] = (\theta_1\hat{\theta}_2)/(\theta_2\hat{\theta}_1) \sim F_{2n_2,2n_1}$. Under H_0, W=$(R_0/(1-R_0))(\hat{\theta}_2/\hat{\theta}_1) \sim F_{2n_2.2n_1}$.

The p-value for this test is $z = 2\min[P_{H_0}(W >$ *w,PH0W<w=2*min *[[1−Fw],F(w)]*, where w is the observed value of test statistic W, and F is the distribution of W under H_0. The p-value of this test indicates how strongly H_0 is supported by the data. A large p-value indicates that R is close to prior estimate R_0 (Tse and Tso, 1996). Thus we use this p-value to form the shrinkage estimator $\tilde{R}_2 = c_2\hat{R}_1 + (1-c_2)R_0$, where $(1-c_2)$ is the p-value of the test.

3. Performance of the estimators

A simulation study is conducted to investigate the performance of the estimators $\tilde{R}_1$ and $\tilde{R}_2$. The nomenclature of our simulations is as follows.

n_1: number of X observations and is taken to be 10 and 30
n_2: number of Y observations and is taken to be 10 and 30
R: the true value of R=p[Y<X] and is taken to be 0.5, 0.6,and 0.8
R_0: The initial estimate of R and is taken to be
0.3,0.4,0.5,0.6,0.7 when R=0.5
0.4,0.5,0.6,0.7,0.8 when R=0.6
0.6,0.7,0.8,0.85,0.9 when R=0.8

Fixing β=2, for each combination of n_1,n_2, R, R_0, 1000 samples were generated for X taking θ=2 and for Y with θ_2=$(1/R_2)$-1. The estimators are calculated and the efficiencies of shrinkage estimators relative to the maximum likelihood estimator are obtained. The relative efficiency is calculated as the ratio of mean square error of the MLE to the mean square error of the shrinkage estimator.

4. Results and Conclusions

From the following table it is observed that shrinkage estimators are more efficient than the maximum likelihood estimator. But the estimator $\tilde{R}_1$ performs better than the estimator $\tilde{R}_2$. In terms of sample sizes, the shrinkage estimators seems to perform better for small sample sizes than the large sample sizes. This is expected, as sample size increases, the precision of ML estimator increases, whereas the shrinkage estimators are still affected by the prior guess R_0 which may be poorly made. our simulation show that the shrinkage estimators, are successful in taking advantage of prior guess. The use of shrinkage estimator is worth considering if available sample size is small.

Table 1. Relative efficiencies of the estimators where R=0.5

n_1	n_2	R_0	RE1	RE2
10	10	0.3	2.8879	1.0017
10	10	0.4	8.0272	1.0296
10	10	0.5	16.0253	1.1737
10	10	0.6	2.7767	1.1835
10	10	0.7	0.8433	1.0002
10	30	0.3	4.8745	0.9996
10	30	0.4	11.968	1.0055
10	30	0.5	16.2556	1.1617
10	30	0.6	2.5643	1.1229
10	30	0.7	0.7598	0.9604
30	10	0.3	2.4962	1.0013
30	10	0.4	5.3945	1.0211
30	10	0.5	8.0272	1.1209
30	10	0.6	2.5432	1.0967
30	10	0.7	0.8718	0.9980
30	30	0.3	3.0269	0.9986
30	30	0.4	6.2861	1.0009
30	30	0.5	9.1161	1.0259
30	30	0.6	2.3367	1.0043
30	30	0.7	0.7413	0.9595

Table:2 Relative efficiencies of the estimators where R=0.6

n_1	n_2	R_0	RE1	RE2
10	10	0.4	2.8502	1.0022
10	10	0.5	6.7174	1.0243
10	10	0.6	10.5170	1.1651
10	10	0.7	2.2350	1.1770
10	10	0.8	0.6983	0.9446
10	30	0.4	4.4554	1.0000
10	30	0.5	9.8900	1.0012
10	30	0.6	11.4739	1.0734
10	30	0.7	2.0730	1.1108
10	30	0.8	0.6103	0.9271
30	10	0.4	2.2647	0.9980
30	10	0.5	4.1654	1.0328
30	10	0.6	5.6120	1.1319
30	10	0.7	2.0312	1.1103
30	10	0.8	0.6928	0.9445
30	30	0.4	2.6305	0.9988
30	30	0.5	4.8015	1.0080
30	30	0.6	6.5371	1.0057
30	30	0.7	1.9830	1.0249
30	30	0.8	0.6145	0.8944

Table:3 Relative efficiencies of the estimators where R=0.8

n_1	n_2	R_0	RE1	RE2
10	10	0.6	1.7065	1.0008
10	10	0.7	3.2715	1.0159
10	10	0.8	7.2788	1.1690
10	10	0.85	2.5709	1.1778
10	10	0.9	0.8517	1.0037
10	30	0.6	3.1512	0.9978
10	30	0.7	4.7595	1.0002
10	30	0.8	7.2358	1.0996
10	30	0.85	2.4804	1.1184
10	30	0.9	0.8188	0.9725
30	10	0.6	1.3380	0.9931
30	10	0.7	2.1688	1.0134
30	10	0.8	4.1213	1.1670
30	10	0.85	2.2215	1.1193
30	10	0.9	0.8462	1.0011
30	30	0.6	1.6695	0.9909
30	30	0.7	2.4007	0.9998
30	30	0.8	4.2925	1.0712
30	30	0.85	2.2277	1.0370
30	30	0.9	0.8453	0.9551

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