International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 3 Issue 3, March - 2014

Solution Second – Order Partial Differential Equation By Using Arbitrary Condition

Hayder Jabbar Abood (1) , Mushtaq Shakir Al-Shehani (2) , Oday Hatem Jalel (3)
(1) Department of Mathematics , College of Education for Pure Sciences , Babylon University , Babylon , Iraq .
(2)Vice Chancellor For Administrative Affairs , University of Babylon, Babylon , Iraq .
(3) Department of Mathematics , College of Education for Pure Sciences , Babylon University , Babylon , Iraq .

***Abstract* : We study the development of Burgers' equation by using the function depending on the time (t), and we prove the existence and uniqueness solution , also given some applications .**

I. Introduction

In [1] consider the problem of stabilization of the inviscid Burgers partial differential equation (PDE) using boundary actuation ,and a solution to the problem using a Lyapunov approach and proved that the inviscid Burgers equation is stabilizable around a constant uniform state under an appropriate boundary control . In [2], In this paper we study the generalized Burgers equation $u_t + (u^2/2)_x = f(t)u_{xx}$, where f (t)>0 for t > 0 . The existence and uniqueness of classical solutions to the initial value problem of the generalized Burgers equation with rough initial data belonging to L^{∞} (R), as well it is obtained the decay rates of u in L^p norm are algebra order for $p\in [1,\infty]$. Burgers' equation with uncertain initial and boundary conditions is approximated using a polynomial chaos expansion approach where the solution is represented as a series of stochastic, orthogonal polynomials, even though the analytical solution is smooth, a number of discontinuities emerge in the truncated system , in [3] . In [4] , a method for the solution of Burgers' equation is described ,the marker method relies on the definition of a convective field associated with the underlying partial differential equation ; the information about the approximate solution is associated with the response of an ensemble of markers to this convective field . An analysis of dispersive / dissipative features of the difference schemes used for simulations of the non-linear Burgers' equation is developed based on the travelling wave asymptotic solutions of its differential approximation , It is showed that these particular solutions describe well deviations in the shock profile even outside the formal applicability of the asymptotic expansions, namely for shocks of moderate amplitudes , in [5] .In [6] , concerned with an interesting numerical anomaly associated with steady state solutions for the viscous Burgers' equation , also we considered Burgers' equation on the interval (0, 1) with Neumann boundary conditions , they showed that even for moderate values of the viscosity and for certain initial conditions, numerical solutions approach non - constant shock type stationary solutions .In [7] , consider Burgers' equation with a time delay , by using the Liapunov function method , they showed that the delayed Burgers' equation is exponentially stable if the delay parameter is sufficiently small , also give an explicit estimate of the delay parameter in terms of the viscosity and initial conditions, which indicates that the delay parameter tends to zero if the initial states tend to infinity or the viscosity tends to zero . The 1D Burgers equation is used as a toy model to mimick the resulting behaviour of numerical schemes when replacing a conservation law by a form which is equivalent for smooth solutions, such as the total energy by the internal energy balance in the Euler equations , if the initial Burgers equation is replaced by a balance equation for one of its entropies (the square of the unknown) and discredited by a standard scheme , the numerical solution converges, as expected, to a function which is not a weak solution to the initial problem , in [8] . In [9] , they proved that the temperature distribution in the limit one – dimensional rod with time – averaged sources of heat is the uniform asymptotic approximation of the temperature distribution in the initial problem in an arbitrary sub domain of the plane rod and in an arbitrary time interval , which are located at a positive distance from the ends of the rod and the initial time instance , respectively ; of course ,the temperature in the one – dimensional rod , which is a function of the longitudinal coordinate x and the time t , is identified with the function of (x , y ,t) , which is independent of the transversal coordinate y of the plane rod .In [10] , they are obtained an asymptotic expansion, containing regular boundary corner functions in the small parameter, for the solution of a second order partial differential equation, they are constructed the asymptotic expansion $u_n(x, t, \varepsilon)$ for the modified problem and prove it is the unique solution, also they have proved that the solution is valid uniformly in the domain, and the asymptotic approximation is within $O(\varepsilon^{n+1})$. In [11], they are studied the development wave equation with some conditions and proving the existence and uniqueness solution by using the reflection method .In [12], they are studied an modification of an initial – boundary – value problem in the critical case for the heat – conduction equation in a thin domain , they are justify asymptotic expansions of the solution of the problems with respect to a small parameter ε>0 , also proved that the solution is uniform in the domain and the asymptotic approximation is within $O(\varepsilon^{n+1})$. In [13], constructed asymptotic first – order solution of a partial differential equation with small parameter , and have proven the solution is unique and uniform in the domain and, further , the asymptotic approximation is within. In this work , we proving the existence and uniqueness of second – order partial differential function by using arbitrary conditions .

II. STATEMENT OF THE PROBLEM :

Consider the second – order partial differential equation definitions of the following form :

$$\frac{\partial u(x,t)}{\partial t} + g(t)\frac{\partial u(x,t)}{\partial x} = k\frac{\partial^2 u(x,t)}{\partial x^2} \qquad (1.2)$$

$$x \in R \;\; ,t > 0 \, , \; u(x,0) = f(x) \quad , \quad x \in R$$

where g(t)and f(x) are continuous functions define in Ω , Ω={(x,t),

$x \in R, t > 0$} also we suppose the following condition is valid , for given

$x \in R$ *If there exist* τ *and* $\xi_1 < \xi_2$ *such that*

$$u_0(\xi_j) = \frac{x-\xi_j}{\tau} \quad , j = 1,2 \qquad (2.2)$$

$$then\ u_k(x,t) \sim u_0(\xi_1), where\ \ t > \tau\ \ and\ u_k(x,t) \sim u_0(\xi_2)\ ,$$
$$where\ \ t < \tau, as\ \varepsilon \to 0\,.$$

We can change the equation by use the new variables

$$u(x,t) = -k^2 \frac{V_x(x,t)}{V(x,t)} \qquad (3.2)$$

we have

$$u_x(x,t) = -k^2\,\frac{VV_{xx} - V_xV_x}{V^2} = -h(t)k^2\,\frac{V_{xx}}{V} + h(t)k^2\frac{V_x^2}{V^2}$$

$$u_{xx}(x,t) = -k^2\frac{VV_{xxx} - V_{xx}V_x}{V^2} + k^2\frac{V^2\,2V_x\,V_{xx} - V_x^2\,2V}{V^4}$$

$$= -k^2\frac{V_{xxx}}{V} + k^2\frac{V_{xx}V_x}{V^2} + 2k^2\frac{V_xV_{xx}}{V^2} - 2k^2\frac{V_x^2}{V^3}$$

$$= -k^2\,\frac{V_{xxx}}{V} + 3k^2\,\frac{V_{xx}V_x}{V^2} - 2k^2\frac{V_x^2}{V^3}$$

$$u_t(x,t) = -k^2\,\frac{V\,V_{xt} - V_x\,V_t}{V^2} = -k^2\,\frac{V_{xt}}{V} + k^2\,\frac{V_x\,V_t}{V^2}$$

Compensate derivatives of x and t in the equation (1.2) , we get :

$$\left[-k^2\,\frac{V_{xt}}{V} + k^2\,\frac{V_x\,V_t}{V^2}\right] - \left[h(t)k^2\,\frac{V_{xx}}{V} - h(t)k^2\frac{V_x^2}{V^2}\right] = k\left[-k^2\,\frac{V_{xxx}}{V} + 3k^2\,\frac{V_{xx}V_x}{k^2} - 2k^2\frac{V_x^2}{V^3}\right]$$

$$-k^2\,\frac{V_{xt}}{V} - h(t)k^2\,\frac{V_{xx}}{V} + h(t)k^2\,\frac{V_x^2}{V^2} + 2k^3\,\frac{V_x^2}{V^3} - 3k^3\frac{V_{xx}V_x}{V^2} +$$

$$k^2\frac{V_xV_t}{V^2}+\ k^3\frac{V_{xxx}}{V}=0$$

$$\left[-h(t)k^2\frac{V_{xx}}{V}-3k^3\frac{V_{xx}\ V_x}{V^2}\right]+\left[h(t)k^2\frac{V_x^2}{V^2}+2k^3\frac{V_x^2}{k^3}\right]-$$

$$\left[k^2\frac{V_{xt}}{V}-k^2\frac{V_xV_t}{V^2}-\ k^3\ \frac{V_{xxx}}{V}\right]=\ 0$$

$$\left[-h(t)-3k\frac{V_x}{V}\right]+\left[h(t)+2k\frac{1}{V}\right]-\left[V_{xt}-\frac{V_xV_t}{V}-kV_{xxx}\right]=0$$

$$\left[h(t)-3k\frac{V_x}{V}+h(t)+2k\frac{1}{V}\right]-\left[V_{xt}-\frac{V_xV_t}{V}-kV_{xxx}\right]=0$$

$$[-kV_x-V_{xt}]+\left[\frac{V_xV_t}{V}-kV_{xxx}\right]=0$$

$$[-kV-V_t]_x+\left[\frac{V_xV_t}{V}-kV_{xxx}\right]=0$$

If we suppose u =u(x , t) , then we solving the equation (1.2), is :

$$f(x)=\ -k^2\frac{V_x(x,t)}{V(x,t)}=-k^2\frac{dv}{dx}\frac{1}{v}$$

$$\frac{f(x)}{dv}=-k^2\frac{1}{vdx}\rightarrow\int\frac{dv}{f(x)}=\frac{1}{-k^2}\int vdx$$

$$lnv_1=\frac{1}{-k^2}\int_0^x f(s)ds\ \rightarrow\ e^{ln\,v_1}=\ e^{\frac{-1}{k^2}\int_0^x f(s)ds}$$

$$V_1(x,0)=\ P(x)=\ Ce^{-F(x)}\qquad where\quad F(x)=\tfrac{1}{k^2}\int_0^x f(s)ds\ ,$$

and

$$V(x,t)=\frac{1}{k\sqrt{\pi t}}\ \int_R\ e^{\frac{-1}{k^2}\int_0^\delta f(s)ds-\frac{1}{k^2t}(x-\delta)^2}\,d\delta\ ,$$

$$=\frac{1}{\sqrt{k^2\pi t}}\int_R\ e^{\frac{-1}{k^2}M(x,t\,,\delta)}\,d\delta\ ,$$

where

$$M(x,t\,,\delta)=\frac{(x-\ \delta)^2}{t}+\int_0^\delta f(s)ds\ ,$$

and

$$u(x,t)=\frac{\int_R\frac{(x-\delta)}{t}e^{\frac{-1}{k^2}M(x,t,\delta)}\ \ d\delta}{H(t)}\tag{4.2}$$

which agrees with what was found by the first approach for solving the Development Burger's equation. from (4.2) we get

$$u(x,t) \sim \frac{(x-\delta)}{t} = u_0(\delta)$$

Can rewrite the solution to be asymptotically

$$u = u_0(\delta)$$

$$x = \delta + tu_0(\delta) \quad \rightarrow \delta = x - tu_0 \quad \rightarrow \delta = x - tu$$

Since $\boldsymbol{u = u_0(\delta)}$ then we get :

$$u = u_0(x - tu) \tag{5.2}$$

The equation (5.2) is exactly the solution of the problem

$$\frac{\partial u(x,t)}{\partial t} + g(t)\frac{\partial u(x,t)}{\partial x} = 0 \tag{6.2}$$

$$x \in R \ , t > 0 \, , \qquad u(x,0) = f(x) \quad , \quad x \in R \ .$$

we found the equation (6.2) by the method of characteristics, the solution of equation (5.2) is smooth for t, if the function u_0 (x) is smooth function and differentiate for x , then we have

$$u_x = u_0'(\delta)(1 - tu_x) \ ,$$

$$u_x = \frac{u_0'(\delta)}{1 + u_0'(\delta)t} \ ,$$

where

$$1 + u_0'(\delta)t \neq 0 \quad ,$$

If suppose u' (x) < 0 for every x , then $u_x = \infty$, if $t = \frac{-1}{u_0'(\delta)}$ is the first instant T_O when $u_x = \infty$, known as gradient catastrophe, corresponds to a S_o where $u'_0(x)$ has a minimum

$$T_0 = \frac{-1}{u_0'(\delta_0)} \quad , \qquad u_0''(s_0) = 0$$

III. WE CAN STUDY SOME APPLICATION ABOUT EQUATION (1.2):

A. Find the instant of gradient catastrophe for the problem

$$\frac{\partial u(x,t)}{\partial t} + g(t)\frac{\partial u(x,t)}{\partial x} = u$$

$$x \in R \ , t > 0 \, , \qquad u(x,0) = x^2 + 1 \ , \quad x \in R$$

Solution: The solution of the problem

$$u(x,t) = (\, x^2 - u(x,t)t) + 1$$

For the function $u_0(x) = \; x^2 + 1$

$$u_0' = 1 \qquad now\ at \qquad x = 0$$

$$min\ u_0' = max\ (1)$$

$$min\ u_0'(x) = (\min 1) = 1$$

Because $\quad x^2 + 1 = 0 + 1 = 1$

Note that

$$\lim_{x \to 0}(x^2 + 1) = 1 + H(x)$$

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

The graphs of the functions $x^2 + 1$

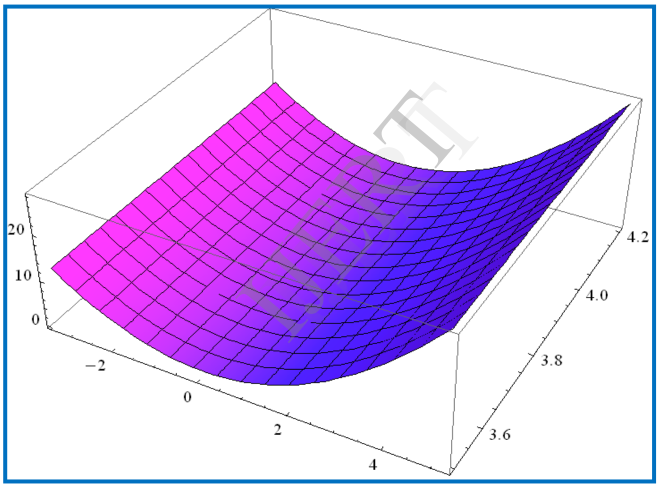


Figure (1.3)

$$\text{Plot3D}[1 + x^2, \{x, -3.4, 5.2\}, \{y, 3.5, 4.2\}$$

B. *Solve the initial value problem foe u(x , t) at time t > 0in terms of t and a characteristic variable*

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{t}} + \boldsymbol{t}\frac{\partial \boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{x}} = \mathbf{1} \quad with$$

u(x,0) =u_0*(x)=* $1 - \frac{1}{2} tanhx$, *do characteristic cross for any t >0 and if so where and when ?*

Solution:

$$\frac{\partial u(x,t)}{\partial t} + t\frac{\partial u(x,t)}{\partial x} = 1 \qquad , \qquad \frac{dt}{1} = \frac{dx}{t} = \frac{du}{1} \quad , \quad so \qquad \frac{du}{dt} = 1$$

$$u = \; A(k) + t \qquad and \qquad \frac{dx}{dt} = t \qquad , \qquad \frac{dx}{dt} = t = u - A(k)$$

$$dx = udt - A(k)\,dt \quad \rightarrow \quad x = ut - A(k)t + k$$

hence in terme of t and k

$$u = A(k) + t \quad with \quad x = ut - A(k)t + k \quad at \;\; t = 0 \quad \rightarrow x = k$$

$$and \quad u = A(x) = u_0(x) = 1 - \frac{1}{2}\tanh x$$

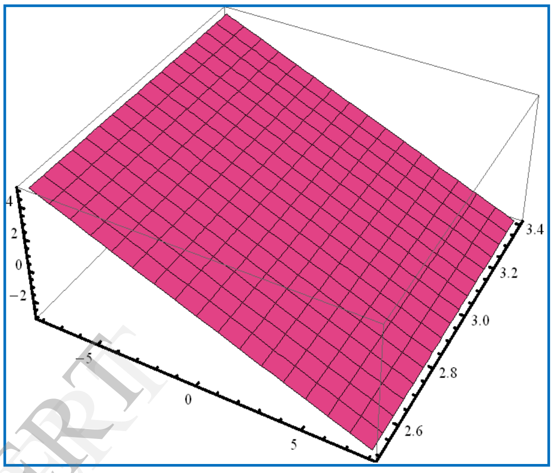
$$so \quad u = 1 - \frac{1}{2}\tanh k + t \quad with \quad x = ut - t[1 - \frac{1}{2}\tanh k] + k$$

$$if \qquad 0 = x_k = -1 - \frac{1}{2}\,t\,sech^2 k$$

$$that\;\; is\; when \;\; t = \frac{2}{sech^2 k} \quad for\; any \quad k \in R \quad at \quad x = k + ut - \frac{2 - \tanh\, k}{sech^2 k}$$

Figure (2.3)

$$\text{Plot3D}[1 - \frac{\text{tanhx}}{2}, \{\text{tanhx}, -8{,}8\}, \{x, 2.5{,}3.4\}$$



IV. PROCEDURE OF SOLVING THE PROBLEM :

Consider the following problem :

$$\frac{\partial u(x,t)}{\partial t} + g(t)\frac{\partial u(x,t)}{\partial x} = 0 \qquad (1.4)$$

$$x \in R\,, t > 0 \quad , u(x,0) = f(x) = u_0(x) \quad , \quad x \in R$$

Which is a limit case of Development Burgers' equation as $\varepsilon \rightarrow 0$.

Definition 1.4 :

Assume $u_0(x) \in L^1(R)$, $A\; function\; u(x,t) \in L^2(R \times [0,\infty])is\; a\; weak\; solution\; of (1.2) \quad iff$

$$\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + h(t)u\varphi_x)dxdt + \int_{-\infty}^\infty u_0(x)\varphi(x,0)dx = 0$$

$$for\; every\; test\; function \qquad \varphi \in C_0^1\,(R \times [0,\infty])\,.$$

Now we need the following Proposition:

Proposition 1.4 : [1]

Let u (x, t) be a smooth solution of the problem $\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = 0, \quad x \in R\,, t > 0$.

$u(x,0) = u_0(x)$ $, x \in R$. Then u (x ,t) is a weak solution of the problem.

Proof:

Obviously $u(x,t) \in L^1_{loc}(R \times [0,\infty])$. $Let\ \rho(x,t) \in C^1_0(R \times [0,\infty])$

and $supp\rho \subseteq [-T,T] \times [0,T]$ Multiplying $u_t + au_x = 0\ by\ \rho$ integrating in $[-T,T] \times [0,T]$ and using $\rho(\pm T,t) = \rho(x,T) = 0$ we obtain :

$$0 = \int_0^T \int_{-T}^T \left(\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x}\right) \rho dxdt$$

$$= \int_0^T \int_{-T}^T ((u\rho)_t + a(u\rho)_x - u(\rho_t + a\rho_x)dxdt$$

$$= \int_0^T \int_{-T}^T (u\rho)_t dtdx - \int_0^T \int_{-T}^T u(\rho_t + a\rho_x)dxdt$$

$$= -(\int_{-T}^T u_0(x)\rho(x,0)dx + \int_0^T \int_{-T}^T u(\rho_t + a\rho_x)dxdt \qquad \blacksquare$$

Theorem 1.4

For a given function $u \in W^{2,1}_{loc}$ (Ω) *the regularizations* $J_\varepsilon u$ *tend to* u *in* $W^{2,1}$ (k) *for every compact* $k \subset \Omega$, i ,e $\| J_\varepsilon u - u\|\, W^{2,1}(k) \to 0$ as $\varepsilon \to 0$. [1]

For the theorem (1.4) and proposition (1.4) the following theorem is holes :

Theorem 2.4

Let $u \in C^1$ $(R \times [0,\infty])$ be a smooth solution of the equation $u_t + h(t)u_x = 0$ and a weak solution of the problem (1.2). If $u_0(x)$ is continuous at a point x_0 , then $u(x_0 , 0) = u_0(x)$.

Proof :

Let $\varphi(x,t) \in C^1_0(R \times [0,\infty])$. As in proposition (1.4) we are led to

$$\int_{-\infty}^{\infty} \big(u(x,0) - u_0(x)\big)\varphi(x,0)dx = 0$$

Suppose $u(x_0,0) > u_0(x_0)$. By continuity there exist a neighborhood U such that

$$u(x,0) > u_0(x) , \quad x \in U$$

Take $\varphi(x,t) \in C_0^1(R \times [0,\infty])$ such that

$$supp\varphi(x,0) = [a,b] \subset U \ , \varphi(x) > 0 \ , x \in (a,b) ,$$

$$\varphi(a) = \varphi(b) = 0$$

Then

$$\int_{-\infty}^{\infty} \big(u(x,0) - u_0(x)\big) \, \varphi(x,0)dx$$

$$= \int_{-\infty}^{\infty} \big(u(x,0) - u_0(x)\big)\varphi(x,0)dx > 0$$

Which is a contradiction , similarly $u(x_0 , 0) < u_0(x_0)$ is impossible . then $u(x_0 , 0) = u_0(x_0)$ ■

Now let us consider the initial data

$$u_0(x) = \begin{cases} u_l & x < 0 \\ u_\tau & x > 0 \end{cases} \qquad (2.4)$$

where u_l and u_τ are constant .

The two cases $u_l > u_\tau$ *and* $u_l < u_\tau$ are quite different with respect to the solvability of problem(1.4) . It can be proved that if $u_l > u_\tau$ then the weak solution is unique, while if $u_l < u_\tau$ then there exist infinitely many solutions.

V. STUDYING THE FOLLOWING CASES FOR PROBLEM (1.4) :

<u>Case I</u> : $u_l > u_\tau$ then we have the following problem

$$u_t(x,t) + h(t)u_x(x,t) = ku_{xx}(x,t)$$

$$x \in R \ , t > 0 , \qquad u(x,0) = f(x) = u_0(x) \ , \ x \in R$$

If $u_l > u_\tau$ we are in a situation to apply the theorem (1.4) , Let x > 0 be fixed and

$$m = \frac{u_l + u_\tau}{2}$$

The instant τ of the theorem (2.4)and by using the condition (2.2) is determined by the slope k of the straight line through the points (x,0) and (0,m)

$$k = -\frac{1}{\tau} = -\frac{m}{x} \ ,$$

then

$$\boldsymbol{\tau = \frac{x}{m}} \ ,$$

and

$$u(x,t) \sim \begin{cases} u_l & x < mt \\ u_\tau & x > mt \end{cases} \qquad as \quad \varepsilon \to 0$$

The unique solution of problem (1.4) is known as a shock wave , while $\boldsymbol{m = (u_l + u_\tau)/2}$ is a shock speed the speed at which the discontinuity of the solution travels .

Theorem 1.5 :

Let the function $$u(x,t) = \begin{cases} u_l & x < mt \\ u_\tau & x > mt \end{cases} \tag{1.5}$$

Is a weak solution of the problem (1.4) with initial data (2.4) , where $u_l > u_\tau$, $m = \frac{u_l + u_\tau}{2}$

Proof :

Let $\varphi(x,t) \in C_0^1(R \times [0,\infty])$. Denote for simplicity

$$A = \int_0^\infty \int_{-\infty}^\infty (u\varphi_t + h(t)u\varphi_x)dxdt$$

$$B = -\int_{-\infty}^\infty u(x,0)\,\varphi(x,0)dx$$

We have

$$A = \int_0^\infty \int_{-\infty}^{mt} (\varphi_t u_l + h(t)\varphi_x u_l)dxdt + \int_0^\infty \int_{mt}^\infty (\varphi_t u_\tau + h(t)\,\varphi_x u_\tau)dxdt$$

$$A_1 = \int_0^\infty \int_{-\infty}^{mt} (\varphi_t u_l + h(t)\varphi_x u_l)dxdt$$

$$= u_l \int_0^\infty (\int_{-\infty}^{mt} \varphi_t dx)dt + u_l \int_0^\infty h(t)(\int_{-\infty}^{mt} \varphi_x dx)dt$$

by

$$\int_{-\infty}^{mt} \varphi_t(x,t)dx = \frac{d}{dt}\int_{-\infty}^{mt} \varphi(x,t)dx - \varphi(mt,t)m \quad ,$$

$$\int_0^\infty \int_{-\infty}^{mt} \varphi_t(x,t)dxdt = \int_0^\infty (\frac{d}{dt}\int_{-\infty}^{mt} \varphi(x,t)dx) - \varphi(mt,t)m)dt \,,$$

$$\int_{0}^{\infty}\left(\frac{d}{dt}\int_{-\infty}^{mt}\varphi(x,t)dx\right)dt = \int_{-\infty}^{\infty}\varphi(x,\infty)dx - \int_{-\infty}^{0}\varphi(x,0)dx = -\int_{-\infty}^{\mathbf{0}}\boldsymbol{\varphi(x,0)dx},$$

then

$$\int_{0}^{\infty}\int_{-\infty}^{mt}\varphi_t(x,t)dxdt = -\int_{-\infty}^{0}\varphi(x,0)\,dx - m\int_{0}^{\infty}\varphi(mt,t)dt\,.$$

And by

$$\int_{-\infty}^{mt}\varphi_x(x,t)dx = \varphi(mt,t) - \varphi(-\infty,t) = \varphi(mt,t) \qquad ,$$

then

$$A_1 = -u_l\left(\int_{-\infty}^{0}\varphi(x,0)dx + m\int_{0}^{\infty}\varphi(mt,t)dt\right) + u_l\int_{0}^{\infty}h(t)\varphi(mt,t)dt\,.$$

Similarly

$$A_2 = \int_{0}^{\infty}\int_{mt}^{\infty}(\varphi_t u_\tau + h(t)\varphi_x u_\tau)dxdt \quad ,$$

$$= u_\tau\int_{0}^{\infty}\left(\int_{mt}^{\infty}\varphi_t dx\right)dt + u_\tau\int_{0}^{\infty}h(t)\left(\int_{mt}^{\infty}\varphi_x dx\right)dt \quad ,$$

$$= -u_\tau\left(\int_{0}^{\infty}\varphi(x,0)dx - m\int_{0}^{\infty}\varphi(mt,t)dt\right) - u_\tau\int_{0}^{\infty}h(t)\varphi(mt,t)dt\,.$$

Because

$$\int_{mt}^{\infty}\varphi_t(x,t)dx = \frac{d}{dt}\int_{mt}^{\infty}\varphi(x,t)dx + \varphi(mt,t)m \quad ,$$

and

$$\int_{mt}^{\infty}\varphi_x(x,t)dx = \varphi(\infty,t) - \varphi(mt,t) = -\varphi(mt,t) \quad ,$$

then $\quad A = A_1 + A_2$

$$-u_l\left(\int_{-\infty}^{0}\varphi(x,0)dx + m\int_{0}^{\infty}\varphi(mt,t)dt\right) + u_l\int_{0}^{\infty}h(t)\varphi(mt,t)dt -$$

$$u_\tau(\int_0^\infty \varphi(x,0)dx - m\int_0^\infty \varphi(mt,t)dt) + u_\tau\int_0^\infty h(t)\varphi(mt,t)dt$$

$$-u_l(\int_0^\infty \varphi(x,0)dx + m\int_0^\infty \varphi(mt,t)dt) - u_\tau(\int_0^\infty \varphi(x,0)dx -$$

$$m\int_0^\infty \varphi(mt,t)dt + u_l\int_0^\infty h(t)\varphi(mt,t)dt + u_\tau\int_0^\infty h(t)\varphi(mt,t)dt$$

$$-u_l\int_0^\infty \varphi(x,0)dx - u_l m\int_0^\infty \varphi(mt,t)dt - u_\tau\int_0^\infty \varphi(x,0)dx$$

$$+u_\tau m\int_0^\infty \varphi(mt,t)dt + u_l\int_0^\infty h(t)\varphi(mt,t)dt + u_\tau\int_0^\infty h(t)\varphi(mt,t)dt$$

$$-u_l\int_0^\infty \varphi(x,0)dx - u_\tau\int_0^\infty \varphi(x,0)dx - u_l m\int_0^\infty \varphi(mt,t)dt + u_\tau m\int_0^\infty \varphi(mt,t)dt\ .$$

$$-\int_0^\infty \varphi(x,0)dx\ (u_l+u_\tau) - m\int_0^\infty \varphi(mt,t)dt\ \ (u_l-u_\tau)$$

$$+\int_0^\infty h(t)\varphi(mt,t)dt\ \ (u_l+u_\tau)\ ,$$

$$= -\int_0^\infty \varphi(x,0)dx\ (u_l+u_\tau) - \int_0^\infty \varphi(mt,t)dt\ [m(u_l-u_\tau) - h(t)(u_l+u_\tau)] = 0 \quad .$$

On the other hand

$$B = -\int_{-\infty}^\infty u(x,0)\,\varphi(x,0)dx \quad ,$$

$$= -\int_{-\infty}^0 u(x,0)\,\varphi(x,0)dx - \int_0^\infty u(x,0)\,\varphi(x,0)dx \quad ,$$

$$= -u_l\int_{-\infty}^0 \varphi(x,0)dx - u_\tau\int_0^\infty \varphi(x,0)dx = 0 \quad .$$

Then we obtain

A = B ■

Case II : If $u_l < u_\tau$ then we have the following problem :

In this case there exist more than one weak solution, we show the following theorem is valid .

Theorem 2.5

Let the function $$u(x,t) = \begin{cases} u_l & x < u_l t \\ x & u_l t \le x \le u_\tau t \\ u_\tau & x > u_\tau t \end{cases}$$

is a weak solution of the problem (1.4)with initial data (2.4) .

Proof : Let $\varphi(x,t) \in \left(R \times [0,\infty)\right)$ for simplicity we take $u_l = -1$ and $u_\tau = 1$ and denote

$$Y = \int_0^\infty \int_{-\infty}^\infty (u\varphi_t + h(t)u\varphi_x)dxdt$$

$$W = -\int_{-\infty}^\infty u(x,0)\varphi(x,0)dx = \int_{-\infty}^0 \varphi(x,0)dx - \int_0^\infty \varphi(x,0)dx \ .$$

Where $u_l = -1$ and $u_\tau = 1$

the function x satisfies the equation $\frac{\partial u(x,t)}{\partial t} + g(t)\frac{\partial u(x,t)}{\partial x} = 0$

we have

$$Y = -\int_0^\infty \int_{-\infty}^{-t} (\varphi_t(x,t) + h(t)\varphi_x(x,t))dxdt$$

$$+ \int_0^\infty \int_{-t}^{t} (x\,\varphi_t(x,t) + h(t)\,x\,\varphi_x(x,t))dxdt$$

$$+ \int_0^\infty \int_t^\infty (\varphi_t(x,t) + h(t)\varphi_x(x,t))dxdt = Y_1 + Y_2 + Y_3 \ ,$$

where

$$Y_1 = -\int_0^\infty \int_{-\infty}^{-t} (\varphi_t(x,t) + h(t)\varphi_x(x,t))dxdt \ ,$$

$$= \int_0^\infty \left(\frac{d}{dt}\left(\int_{-\infty}^{-t} \varphi(x,t)dx\right) - \varphi(-t,t)(-1)\right)dt \ ,$$

by

$$\int_{0}^{\infty}(\frac{d}{dt}\int_{-\infty}^{-t}\varphi(x,t)dx)dt = \int_{-\infty}^{\infty}\varphi(x,\infty)dx - \int_{-\infty}^{0}\varphi(x,0)dx = \int_{-\infty}^{0}\varphi(x,0)dx$$

and

$$\int_{0}^{\infty}\int_{-\infty}^{-t}(h(t)\,\varphi_x(x,t)dxdt\,) = \int_{0}^{\infty}h(t)\int_{-\infty}^{-t}\varphi_x dxdt\ ,$$

by

$$\int_{-\infty}^{-t}\varphi_x(x,t)dx = \varphi(-t,t) - \varphi(-\infty,t) = \varphi(-t,t)\,,$$

then

$$-\int_{0}^{\infty}\int_{-\infty}^{-t}(\varphi_t(x,t) + h(t)\varphi_x(x,t)dxdt\,),$$

$$= \int_{-\infty}^{0}\varphi(x,0)dx - \int_{0}^{\infty}\varphi(-t,t)dt - \int_{0}^{\infty}h(t)\varphi(-t,.t)dt\,.$$

$$Y_2 = \int_{0}^{\infty}\int_{-t}^{t}(\,x\,\varphi_t(x,t) + \ h(t)\,x\,\varphi_x(x,t))dxdt\ \ ,$$

Y_2 has a singularity at 0

$$Y_2 = \lim_{k\to 0} Y_{2,k}\ \ ,$$

where

$$Y_2 = \int_{k}^{\infty}\int_{-t}^{t}(\,x\,\varphi_t(x,t) + \ h(t)\,x\,\varphi_x(x,t))dxdt\ ,$$

$$= \int_{k}^{\infty}(\int_{-t}^{t}x\varphi_t(x,t)dx)dt + \int_{k}^{\infty}(h(t)\int_{-t}^{t}x\varphi_x(x,t)dx)dt$$

we have

$$\int_{-t}^{t}(x\,\varphi)_t dx = \frac{d}{dt}\int_{-t}^{t}(x\varphi)dx + \varphi(t,t) + \varphi(-t,t)\ \ ,$$

and

$$\int_{-t}^{t} (x\varphi)_x = \varphi(t,t) + \varphi(-t,t) \quad ,$$

so

$$\int_{k}^{\infty} (h(t) \int_{-t}^{t} x\varphi_x(x,t)dx)dt \ = h(t)\int_{k}^{\infty} \varphi(t,t)dt + \int_{k}^{\infty} h(t)\varphi(-t,t)\, dt \quad ,$$

we take the limit as $k \to 0$

$$\lim_{k\to 0} \int_{-k}^{k} x\varphi(x,k)dx = 0$$

then

$$Y_2 = \int_{0}^{\infty} \varphi(t,t)dt + \int_{0}^{\infty} \varphi(-t,t)dt + h(t)\int_{k}^{\infty} \varphi(t,t)dt + \int_{k}^{\infty} h(t)\varphi(-t,t)\, dt \quad .$$

Similarly

$$Y_3 = \int_{0}^{\infty}\int_{t}^{\infty} \big(\varphi_t(x,t) + h(t)\varphi_x(x,t)\big)dxdt$$

$$= -\int_{0}^{\infty} \varphi(x,0)dx - \int_{0}^{\infty} \varphi(t,t)dt - \int_{0}^{\infty} h(t)\varphi(t,t)\, dt \ .$$

Finally

$Y = Y_1 + Y_2 + Y_3$

$$= \int_{-\infty}^{0} \varphi(x,0)dx - \int_{0}^{\infty} \varphi(-t,t)dt - \int_{0}^{\infty} h(t)\varphi(-t,t)dt$$

$$+ \int_{0}^{\infty} \varphi(t,t)dt + \int_{0}^{\infty} \varphi(-t,t)dt + \int_{k}^{\infty} h(t)\varphi(t,t)dt + \int_{k}^{\infty} h(t)\varphi(-t,t)\, dt$$

$$- \int_{0}^{\infty} \varphi(x,0)dx - \int_{0}^{\infty} \varphi(t,t)dt - \int_{0}^{\infty} h(t)\varphi(t,t)\, dt \ = \int_{-\infty}^{0} \varphi(x,0)dx - \int_{0}^{\infty} \varphi(x,0) = W$$

The proof is complete .

VI. We have some applications about the case II :

1.6 : we solve the initial value problem for u(x,t) , t >0 in terms of t and a characteristic variable

$$\frac{\partial u(x,t)}{\partial t} + g(t)\frac{\partial u(x,t)}{\partial x} = 0 \qquad \text{with}$$

$$u(x,0) = u_0(x) = \begin{cases} -2 & , \quad x \le -1 \\ 2x^2 & , \quad -1 \le x \le 1 \\ 2 & , \quad x \ge 1 \end{cases}$$

$$\frac{\partial u}{\partial t} = h(t)\frac{\partial u}{\partial x} \quad , If \quad \partial t = \frac{\partial x}{h(t)} = \frac{\partial u}{0} \quad \to so \quad \frac{\partial u}{\partial t} = 0 \quad \to u = B(L)$$

$$and \quad \frac{\partial x}{\partial t} = h(t) \quad then \quad x = H(t) + L$$

hence in terms of t and L , $u = B(L)$ *with* $x = H(t) + L$ *then* $at\ t = 0 \ \to \ x = L$,

and $$u = B(L) = u_0(x) = \begin{cases} 2 & for \quad |x| \ge 1 \quad , \\ 2x^2 & for \quad |x| \le 1 \quad , \end{cases}$$

so $$u = \begin{cases} 2 & for \quad |L| \ge 1 \quad , \\ 2L^2 & for \quad |L| \le 1 \quad , \end{cases}$$

with $$x = L + H(t) \times \begin{cases} 2 & for \quad |L| \ge 1 \quad , \\ |2L^2| & for \quad |L| \le 1 \quad , \end{cases}$$

we have ,

$$0 = x_L = 4L + H(t) \times \begin{cases} 0 & |L| > 1 \quad , \\ 4L & 0 < L < 1 \quad , \\ -4L & -1 < L < 0 \quad . \end{cases}$$

That is when at t = 0 $for \quad -1 < L < 0 \quad at \quad x = L + H(t) = 0$

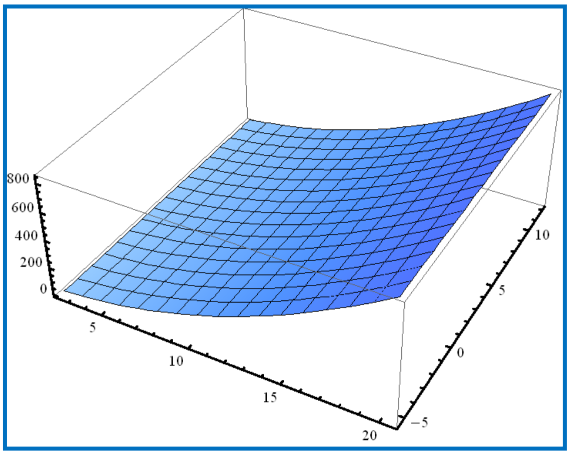


Figure (1.6)

Plot 3D$[2x^2, \{x, 20.5, 2.5\}, \{t, -5.4, 12\}]$

REFERENCES :

1- Ioannis P. Stavroukis and stepan A. Tersian , Partial Differential Equations , world scientific, pp135 , 2003 .
2- Jinghua Wang , Hui Zhang , Existence and decay rates of solutions to the generalized Burgers equation , Journal of mathematical analysis and applications , pp 213–235 , 2003.
3- Per Pettersson , Qaisar Abbasb, Gianluca Iaccarino and Jan Nordstr¨omb , Efficiency of shock capturing schemes for Burgers' equation with boundary uncertainty , Seventh South African Conference on Computational and Applied Mechanics , 2010 .
4- Jerome L. V. Lewandowski , Solution of Burgers, equation using the marker method , International journal of numerical analysis and modeling , Volume 3 , Number 1 , pp 80-93 , 2006 .
5- A.V. Porubov , D. Bouche and G.Bonnaud , Description of numerical shock profiles of non-linear Burgers' equation by asymptotic solution of its differential approximations , 2003 .
6- J. Burnsy , A. Balogh , D.S. Gilliamz and V.I. Shubov , Numerical Stationary Solutions for a Viscous Burgers' Equation , Journal of Mathematical Systems , Estimation, and Control , Volume 8 , number 2 , pp 1-16 1998 .
7- Weijiu Liu , Asymptotic behavior of solutions of time delayed Burgers' equation , Discrete and continuous dynamical system , Volume 2 , Number 1, 2002 .
8- T. Gallou¨et , R. Herbin , J.-C. Latch´e and T.T. Nguyen , Playing with Burgers's equation , 2011 .
9- V. B. Levenshtam and H. D. Abood , Asymptotic Integration of The Problem on Heat Distribution in A Thin Rod Wirth Rapidly Varying Sources of Heat , Journal of Mathematical Science . Volume 129 , Number 1 , 2005 .
10- Hayder Jabber Abood and Ali Abbas Jabir , Asymptotic Approximation of the Second Order Partial Differential Equation by Using Many Functions , British Journal of Science , Volume 5 , 2012.
11- Hayder Jabber Abood and Ahmed Hadi Hussain , The Existence Solution to the Development Wave Equation With Arbitrary Conditions , Mathematical Theory and Modeling , Volume 3 , Number 4 , 2013.
12- Hayder Jabber Abood, On The Formation of an Asymptotic Solution of the Heat Equation With Small Parameter, European Journal of Scientific Research , Volume 56, Number 4 ,pp471-481 , 2011 .
13- Hayder Jabber Abood, Estimation of Partial Differential Equations Depending on a Small Parameter , American Journal of Scientific Research , pp32-39 , 2011