Some New Inequalities For The Generalized $\epsilon -$ Gamma, Beta And Zeta Functions

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Abstract

In this paper, we establish some properties and inequalities for the ϵ- generalized functions which are ϵ - Gamma function, Beta function and Zeta function and has given some identities which they satisfy. This inequality leads to new inequalities involving the Beta, Gamma and Zeta functions and a large family of functions. The gamma and Beta functions belong to the category of the special transcendental functions and we will see that some mathematical constants are occurring in its study.

Keywords: ϵ- generalized Gamma function , ϵ – Beta function and ϵ - Zeta function.

1. Introduction

The generalized ϵ - Gamma function $\Gamma_\epsilon(x)$ as

$$\Gamma_\epsilon(x) = \lim_{n\to\infty} \frac{n!\epsilon^n (n\epsilon)^{x/_{\epsilon}-1}}{(x)_{n,\epsilon}}, k>0, x \in C - \epsilon Z^- \qquad (1.1)$$

where $(x)_{n,\epsilon}$ is the ϵ- Pochammer symbol and is given by

$$(x)_{n,\epsilon} = x(x+\epsilon)(x+2\epsilon)\dots(x+(n-1)\epsilon), x\in C, \epsilon \in R, n\in N^+ \qquad (1.2)$$

It is obvious that $\Gamma_\epsilon(x) \to \Gamma(x)$ for $\epsilon \to 1$, where $\Gamma(x)$ is known as Gamma function. Also for Re (x) > 0, it holds

$$\Gamma_\epsilon(x) = \int_0^\infty t^{x-1} e^{-t^\epsilon/_\epsilon}\, dt \qquad (1.3)$$

And it follows that

$$\Gamma_\epsilon(x) = \epsilon^{\frac{x}{\epsilon}-1}\Gamma\left(\frac{x}{\epsilon}\right). \qquad (1.4)$$

In this paper [1],[2],[3] introduced the ϵ- Beta function $B_\epsilon(x,y)$ as

$$B_\epsilon(x,y) = \frac{\Gamma_\epsilon(x)\Gamma_\epsilon(y)}{\Gamma_\epsilon(x+y)}, \text{Re}(x) > 0, \text{Re}(y) > 0 \qquad (1.5)$$

And ϵ-Zeta function as

$$\zeta_\epsilon(x,s) = \sum_{\vartheta=0}^\infty \frac{1}{(x+\vartheta\epsilon)^s}, \quad \epsilon, s > 0, s > 1 \qquad (1.6)$$

The function $B_\epsilon(x,y)$ satisfies the equality

$$B_\epsilon(x,y) = \frac{1}{\epsilon}\int_0^1 t^{\frac{x}{\epsilon}-1}(1-t)^{\frac{y}{\epsilon}-1} dt \qquad (1.7)$$

which follows

$$B_\epsilon(x,y) = \frac{1}{k} B\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right). \qquad (1.8)$$

We mention that $\lim_{n\to\infty} B_\epsilon(x,y) \to B(x,y)$ and ϵ- Zeta function is a generalization of Hurwitz Zeta function $\zeta(x,s) = \sum_{\vartheta=0}^\infty \frac{1}{(x+\vartheta\epsilon)^s}$ which is a generalization of the Reimann Zeta function

$\zeta(s) = \sum_{\vartheta=1}^\infty \frac{1}{(\vartheta)^s}$. The motivation to study properties of generalized ϵ –Gamma and ϵ- Beta functions is the fact that $(x)_{n,\epsilon}$ appears in the combinatorics of creation and annihilation operators[3]. Recently M. Mansour [4] determined the ϵ – generalized Gamma function by a combination of some functional equations.

In this paper , we use the definitions of the above generalized functions to prove a formula for $\Gamma_\epsilon(2x)$ which is a generalization of the Legendre duplication formula for $\Gamma(x)$ and to prove inequalities for the function $B_\epsilon(x,y)$, for x,y, ϵ > 0 and x + y $\neq \epsilon$ and the product $\Gamma_\epsilon(x)$ $\Gamma_\epsilon(1-x)$, for 0 < x, ϵ < 1.we also give monotonicity properties for $\psi_\epsilon(x) = \partial_x\psi(\epsilon,x)$ where $\psi(\epsilon,x) = log\Gamma_\epsilon(x)$ and $\zeta_\epsilon(x,s)$ for s $\in N$ and $s \geq 2$.

We mention that using (1.4) the following inequalities hold:

$$\Gamma_\epsilon(a\,\epsilon) = \epsilon^{a-1}\Gamma(a), \epsilon>0, a\in R \qquad (1.9)$$

$$\Gamma_\epsilon(n\,\epsilon) = \epsilon^{n-1}(n-1)!, \ \epsilon>0, n\in N \qquad (1.10)$$

$$\Gamma_\epsilon(\,\epsilon) = 1, \epsilon>0, \qquad (1.11)$$

$$\Gamma_\epsilon((\,2n+1)\tfrac{\epsilon}{2}) = \epsilon^{\frac{2n-1}{2}}\frac{(2n)!\sqrt{\pi}}{2^n n!}, \epsilon>0, n\in N \qquad (1.12)$$

also , using (1.5) and (1.8) the following equalities hold:

$$B_\epsilon(x+\epsilon, y) = \frac{x}{x+y}B_\epsilon(x,y), B_\epsilon(x, y+\epsilon)$$

$$= \frac{y}{x+y}B_\epsilon(x,y) \qquad x, y, \epsilon > 0. \qquad (1.13)$$

$$B_\epsilon(x,\epsilon) = \frac{1}{x} \text{ and } B_\epsilon(\epsilon, y) = \frac{1}{y}, \ \ x, y, \epsilon > 0. \qquad (1.14)$$

$$B_\epsilon(a\epsilon, b\epsilon) = \frac{1}{x}B(a,b), \text{ a,b, } \epsilon > 0 \qquad (1.15)$$

$$B_\epsilon(n\epsilon, n\epsilon) = \frac{1}{\epsilon}\frac{(n-1)!^2}{(2n-1)!}, \ \epsilon > 0, \ n \in N \qquad (1.16)$$

2. The Function $\Gamma_\epsilon(\,x)$

Theorem2.1: let x, ϵ > 0 and $\psi_\epsilon(x)$ be the logarithmic derivative of $\Gamma_\epsilon(\,x)$. Then the function $\psi_\epsilon{}'(x)$ is completely monotonic.

Proof: From (1.4) , we get

$\mathrm{Log}\Gamma_\epsilon(\,x) = \left(\frac{x}{\epsilon}-1\right) log\epsilon + \log\Gamma_\epsilon(\,x/\epsilon)$ or by setting $\psi(\epsilon,x) = \mathrm{Log}\Gamma_\epsilon(\,x)$, we obtain

$$\psi(\epsilon,x) = \left(\frac{x}{\epsilon}-1\right) log\epsilon + \log\Gamma_\epsilon(\,x/\epsilon) \qquad (2.1)$$

We get $,\partial_x\psi(\epsilon,x) = \left(\frac{1}{\epsilon}\right) log\epsilon + \psi(x/\epsilon)$ (2.2)

We remind that $\psi(x/\epsilon) = \partial_x\left(log\Gamma_\epsilon\left(\frac{x}{\epsilon}\right)\right)$. from (2.2) taking the derivative with respect to x, we have

$$\partial_x{}^2\psi = \left(\frac{1}{\epsilon}\right)\psi_\epsilon{}'(x/\epsilon) \qquad (2.3)$$

$$\partial_x{}^3\psi(\epsilon,x) = \left(\frac{1}{\epsilon^2}\right)\psi_\epsilon{}''(x/\epsilon)$$

By induction, we obtain $\partial_x{}^{n+1}\psi(\epsilon,x) = \left(\frac{1}{\epsilon^n}\right)\psi_\epsilon{}^n(x/\epsilon)$

Or if we call $\psi_\epsilon(x) = \partial_x\psi(\epsilon,x)$, then the equation

$$\psi_\epsilon{}^n(x) = \left(\frac{1}{\epsilon^n}\right)\psi_\epsilon{}^n(x/\epsilon) \qquad (2.4)$$

holds. It is known [1] that $\psi_\epsilon{}'(x)$ is completely monotonic for x > 0, so from (2.4) it follows the desired result.

Remark 2.1. (i) From (2.3) it follows that $\Gamma_\epsilon(\,x)$ is logarithmic convex on $(0,\infty)$ which is proved in [2], (ii) Theorem 2.1 is a generalization of the known [1] result that the function $\psi_\epsilon{}'(x)$ is completely monotonic.

Result 2.1. For x > 0 the function $\psi(\epsilon,x) = \log\Gamma_\epsilon(\,x)$ satisfies the differential equation

$$-x^2\epsilon\partial^2{}_\epsilon\psi(\epsilon,x) + 2\epsilon^2\partial_\epsilon\psi(\epsilon,x) + \epsilon^3\partial_\epsilon\psi(\epsilon,x) = -x-\epsilon \qquad (2.5)$$

Proof: From (2.1) taking the first and second derivative of $\psi(\epsilon,x)$ with respect to ϵ, we obtain

$$\partial_\epsilon\psi(\epsilon,x) = \frac{-x}{\epsilon^2}log\epsilon + \frac{x}{\epsilon^2} - \frac{1}{\epsilon} - \frac{1}{\epsilon}\psi(x/\epsilon) \qquad (2.6)$$

$$\partial^2{}_\epsilon\psi(\epsilon,x) = \frac{2x}{\epsilon^3}log\epsilon - \frac{3x}{\epsilon^3} + \frac{1}{\epsilon^2} + \frac{x}{\epsilon^2}\psi(x/\epsilon) + \frac{x}{\epsilon^3}\psi_\epsilon{}'(x/\epsilon) \qquad (2.7)$$

From (2.3), (2.6) and (2.7), we get (2.5)

Theorem2.2: The function $\Gamma_\epsilon(\,x)$ satisfies the equality

$$\Gamma_\epsilon(\,2x) = \sqrt{\frac{\epsilon}{\pi}}\,2^{2\frac{x}{\epsilon}-1}\Gamma_\epsilon(\,x)\Gamma_\epsilon(\,x+k/2) \qquad (2.8)$$

For $x \in C$ with Re(x) > 0.

Proof: From (1.7) it follows that

$$B_\epsilon(x,x) = \frac{1}{\epsilon}\int_0^1 t^{\frac{x}{\epsilon}-1}(1-t)^{\frac{x}{\epsilon}-1}dt$$

Or by setting $t = \frac{1+r}{2}$, $B_\epsilon(x,x) = \frac{2}{\epsilon 2^{2\frac{x}{\epsilon}-1}}\int_0^1(1-r^2)^{\frac{x}{\epsilon}-1}dr$

or by setting $r^2 = u$,

we obtain $B_\epsilon(x,x) = \frac{1}{\epsilon 2^{2\frac{x}{\epsilon}-1}}\int_0^1 u^{\frac{1}{2}-1}(1-u)^{\frac{x}{\epsilon}-1}du = \frac{1}{\epsilon 2^{2\frac{x}{\epsilon}-1}}B\left(\frac{x}{\epsilon},\frac{x}{\epsilon}\right) = \frac{1}{2^{2\frac{x}{\epsilon}-1}}B_\epsilon\left(x,\frac{\epsilon}{2}\right)$ Or

$$B_\epsilon(x,x) = \frac{1}{2^{2\frac{x}{\epsilon}-1}}\frac{\Gamma_\epsilon(\,x)\Gamma_\epsilon\left(\frac{\epsilon}{2}\right)}{\Gamma_\epsilon\left(x+\frac{\epsilon}{2}\right)} \qquad (2.9)$$

from (1.9) for a=1/2 , we get $\Gamma_\epsilon\left(\frac{\epsilon}{2}\right) = \sqrt{\frac{\pi}{\epsilon}}$, since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, from (2.9) and (1.5) , we get the equality (2.8).

Remark 2.3. Theorem 2.2 is a generalization of the legendre duplication formula of Γ(x).

3. The Function $\zeta_\epsilon(x,s)$

Theorem3.1 (i) Let x, $\epsilon > 0$ and s > 1. Then the positive function $\zeta_\epsilon(x,s)$ decreases with respect to x and also decreases with respect to ϵ. (ii) let x > 0 and s > 1. Then the positive function $\zeta_\epsilon(x,s)$ decreases with respect to s for x > 1 and $\epsilon > 0$, $\vartheta \geq 0$ and increases with respect to s for $\vartheta > 0$, $0 < \epsilon < 1/\epsilon$ and $0 < x < 1 - \vartheta\epsilon$.

Proof: From (1.6) we obtain

$$\partial_x \zeta_\epsilon(x,s) = \sum_{\vartheta=0}^{\infty} \frac{1}{(x+\vartheta\epsilon)^{s+1}}, \quad \epsilon, x > 0, s > 1$$

Or

$$\partial_x \zeta_\epsilon(x,s) = -s\, \zeta_\epsilon(x,s+1) \qquad (3.1)$$

$$\partial_x \zeta_\epsilon(x,s) = \sum_{\vartheta=0}^{\infty} \frac{-\vartheta s}{(x+\vartheta\epsilon)^{s+1}} = -s \sum_{\vartheta=1}^{\infty} \frac{\vartheta}{(x+\vartheta\epsilon)^{s+1}} \quad \epsilon, x > 0, s > 1 \qquad (3.2)$$

Then (3.1) and (3.2) , prove the theorem 3.1(i)also the definition(1.6) gives

$$\partial_s \zeta_\epsilon(x,s) = -\sum_{\vartheta=0}^{\infty} \frac{\ln(x+\vartheta\epsilon)}{(x+\vartheta\epsilon)^s} \qquad (3.3)$$

If x >1 then x > 1- $\vartheta\epsilon$, for $\vartheta, \epsilon > 0$ thus ln(x + $\vartheta\epsilon$) > 0 so from (3.3) it follows that the function $\zeta_\epsilon(x,s)$ decreases with s > 1 and if $0 < \epsilon < 1/\vartheta$ and $0 < x < 1-\vartheta\epsilon$ then ln(x + $\vartheta\epsilon$) < 0 from (3.3) it follows that the function $\zeta_\epsilon(x,s)$ increases with s > 1.

Result 3.1: Let x > 0, $\epsilon > 0$ and s > 1. Then the function $\zeta_\epsilon(x,s)$satisfies the identities:

$$\partial^n{}_x \zeta_\epsilon(x,s) = (-1)^n (s)_{n,1}\, \zeta_\epsilon(x,s+n) \qquad (3.4)$$

$$\zeta_\epsilon(x,s) = (-1)^n \frac{\partial^n{}_x \psi(\epsilon,x)}{(n-1)!}, n \geq 2 \qquad (3.5)$$

And $$\zeta_\epsilon(x+\epsilon,s) = \zeta_\epsilon(x,s) - 1/x^s \qquad (3.6)$$

Proof: From (3.1) we obtain

$$\partial^2{}_x \zeta_\epsilon(x,s) = -s\ \partial_x \zeta_\epsilon(x,s+1) = (-1)^2 s(s+1)\ \zeta_\epsilon(x,s+2)$$

Repeating the same , we get (3.4) ,

since s (s+1) …….(s+n-1) = $(s)_{n,1}$.

In[2]it was proved that

$$\partial^2{}_x \psi(\epsilon,x) = \sum_{\vartheta=0}^{\infty} \frac{1}{(x+\vartheta\epsilon)^2} \qquad (3.7)$$

From (1.6) for s + 2 and (3.7), we get

$$\partial^2{}_x \psi(\epsilon,x) = \zeta_\epsilon(x,2) \qquad (3.8)$$

Differentiating (3.7) with respect to x and using (3.1) for s = 2, we get

$$\partial^3{}_x \psi(\epsilon,x) = (-1)^2\, 2\, \zeta_\epsilon(x,3)$$

and $$\partial^4{}_x \psi(\epsilon,x) = (-1)^2 3!\ \zeta_\epsilon(x,4)$$

By induction, we obtain (3.5). The equation (3.6) follows from the definition(1.6), since

$$\zeta_\epsilon(x,s) = \frac{1}{x^s} + \sum_{\vartheta=0}^{\infty} \frac{1}{(x+\epsilon+\vartheta\epsilon)^s} = \frac{1}{x^s} + \zeta_\epsilon(x+\epsilon,s).$$

4. Inequalities For $B_\epsilon(x,y)$, $\Gamma_\epsilon(x)\Gamma_\epsilon(1-x)$

Theorem 4.1: let x,y, $\epsilon > 0$ and x + y $\neq \epsilon$. Then the function $B_\epsilon(x,y)$ satisfies the inequalitie

$$\frac{2^{2-\frac{x+y}{\epsilon}}}{x+y-\epsilon} < B_\epsilon(x,y) < \frac{1-2^{2-\frac{x+y}{\epsilon}}}{x+y-\epsilon} \qquad (4.1)$$

Lemma 4.1: The function B(x,y) satisfies the inequalities

$$\frac{2^{2-(x+y)}}{x+y-1} < B(x,y) < \frac{1-2^{2-(x+y)}}{x+y-1}, \ x,y > 0,\ x+y \neq 1 \qquad (4.2)$$

Proof: The function B(x,y) is defined [1] by the integral

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

Which can be written as

$$B(x,y) = \int_0^{1/2} t^{x-1} (1-t)^{y-1} dt + \int_{1/2}^1 t^{x-1} (1-t)^{y-1} dt \qquad (4.3)$$

If 0 < t < ½ then t < 1- t , so that the following inequalities hold

$$\int_0^{\frac{1}{2}} t^{x+y-2}\, dt < \int_0^{\frac{1}{2}} t^{x-1} (1-t)^{y-1} dt < \int_0^{1/2} (1-t)^{x+y-2}\, dt \qquad (4.4)$$

and if ½ < t < 1 then 1 – t < t, so that the following inequalities hold

$$\int_{1/2}^{1}(1-t)^{x+y-2}\,dt < \int_{1/2}^{1} t^{x-1}\,(1-t)^{y-1}dt < \int_{1/2}^{1} t^{x+y-2}\,dt \qquad (4.5)$$

From (4.3), using the inequalities (4.4) and (4.5) and evaluating the integrals on the left and right side of the above inequalities , we obtain the inequalities (4.2) .

Proof of theorem 4.1:By setting x / ϵ and y/ ϵ, instead of x and y respectively in (4.2) and taking in account the relation (1.8) we get the inequalities (4.1).

Corollary 4.1: Let x,y, ϵ > 0. Then the function $B_\epsilon(x,y)$ satisfies the inequalities:

$$\frac{2^{1-\frac{x+y}{\epsilon}}}{x} < B_\epsilon(x,y) < \frac{1-2^{1-\frac{x+y}{\epsilon}}}{x} \qquad (4.6)$$

Or $$\frac{2^{1-\frac{x+y}{\epsilon}}}{y} < B_\epsilon(x,y) < \frac{1-2^{1-\frac{x+y}{\epsilon}}}{y} \qquad (4.7)$$

Proof: The above inequalities follow from (4.1) by setting x + ϵ (or y + ϵ) instead of x (or y) and taking in account relations (1.13).

Corollary 4.2: Let 0 < x < 1 and 0 < ϵ < 1. Then the following inequalities for the product $\Gamma_\epsilon(x)\Gamma_\epsilon(1-x)$ holds

$$\frac{\left(\frac{2}{\epsilon}\right)^{1-1/\epsilon}\Gamma(1/\epsilon)}{1-x} < \Gamma_\epsilon(x)\Gamma_\epsilon(1-x) < \frac{\left(\frac{2}{\epsilon}\right)^{1-1/\epsilon}\Gamma(1/\epsilon)\left(2^{\frac{1}{\epsilon}-1}-1\right)}{1-x} \qquad (4.8)$$

Proof: By setting y = ϵ + 1 – x instead of y in (4.1) we obtain

$$2^{1-1/\epsilon} < B_\epsilon(x,\epsilon+1-x) < 1-2^{1-1/\epsilon} \qquad (4.9)$$

Using (1.5) the inequalities (4.9) become

$$2^{1-1/\epsilon} < \frac{\Gamma_\epsilon(x)\Gamma_\epsilon(\epsilon+1-x)}{\Gamma_\epsilon(\epsilon+1)} < 1-2^{1-1/\epsilon} \qquad (4.10)$$

From (1.4) we obtain

$$\Gamma_\epsilon(\epsilon+1-x) = (1-x)\Gamma_\epsilon(1-x)$$

and $\Gamma_\epsilon(\epsilon+1) = \Gamma_\epsilon(1) = \epsilon^{\frac{1}{\epsilon}-1}\Gamma_\epsilon(1/\epsilon)$.

From (4.10) using the above equalities we obtain the inequalities (4.8).

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