# Some Properties of Fuzzy Boolean algebra

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## Abstract

*The aim of this article is to continue the study of the behaviors of the fuzzy Boolean algebra formed by the fuzzy subsets of a finite set that has been introduced in [11]. In this article, the atom and the co-atom of those fuzzy Boolean algebras are introduced and their properties are discussed.*

## Keywords

Fuzzy Boolean algebra, subelement, atom, co-atom

## 1. Introduction

In [11], a kind of family of fuzzy subsets of a finite set had been introduced which can form fuzzy Boolean algebra, where the complement operation on the fuzzy subset was redefined. Further, those works had been extended by observing the properties of homomorphism, isomorphism and automorphism of the fuzzy Boolean algebra in [12]. The characteristics of ideals and filters had also been observed in that article.

The set of atoms forms a basis of the Boolean algebra because all elements can be expressed using these elements.

The aim of this article is to continue the study of the behaviors of the fuzzy Boolean algebra formed by the fuzzy subsets of a finite set which was introduced in [11]. In the first section, the definition of atom for a fuzzy Boolean algebra is introduced and then some properties of those atoms are discussed. The next section is concerned with the co-atoms for a fuzzy Boolean algebra and their properties are discussed.

## 2. Preliminaries

This section lists some basic definitions and concepts of Boolean algebra as follows:

Considering $B$ be an arbitrary Boolean algebra and let $p_0$ be an arbitrary element of $B$. Then the set of elements $p$ with $p \le p_0$ or equivalently, the set of all elements of the form $p \wedge p_0$ is called the set of subelements of $p_0$.

Every Boolean algebra has a trivial subelement or subset namely the set $\{0\}$, consisting the bottom element $0$ alone; all other subelements or subsets of $B$ are called non-trivial. Every Boolean algebra $B$ has an improper subset namely $B$ itself; all other subsets are called proper.

An atom of a Boolean algebra is an element which does not have non-trivial proper subelements. This implies that an element $q$ of a Boolean algebra $B$ is an atom if $q \ne 0$ and if there are only two elements $p$ such that $p \le q$, namely $0$ and $q$. Atoms are the elements which covers $0$. Atoms are also the immediate successor of 0.

A co-atom of a Boolean algebra is an element which does not have any proper superset. This implies that an element $c$ of a Boolean algebra $B$ is a co-atom if $c \ne 0$ and if there are only two elements $d$ such that $c \le d$, namely $1$ and $c$. Co-atoms of a Boolean algebra are the elements which are the immediate predecessor of 1.

The main concept of fuzzy Boolean algebra which has been introduced in the article [11] is as follows:

For a finite set $E = \{x_0, x_1, x_2 ..., x_{n-1}\}$ with $n$ elements with the set $M$ of membership values, such that,

$$M = \left\{0, \frac{1}{p}, \frac{2}{p}, \frac{3}{p}, ..., \frac{p-1}{p}, \frac{p}{p} = 1\right\}$$
$$= \{0, h, 2h, 3h, ..., (p-1)h, ph\},$$

where, $h = \frac{1}{p}$ and $p$ is any number

Then for any mapping $E \to \{0, kh\}$, where $1 \le k \le p$, forms a Boolean algebra. This is called fuzzy Boolean algebra, as it is formed by fuzzy subsets. The fuzzy subset consisting of all membership value equal to $'kh'$ is called the universal fuzzy subset and the fuzzy subset consisting all membership value equal to $'0'$ is called the empty fuzzy subset.

Hence, for the mappings $E \to \{0, h\}$, $E \to \{0, 2h\}, ...E \to \{0, ph\}$,

$p$-numbers of Fuzzy Boolean algebras can be obtained which are denoted by $B_1, B_2, B_3,...B_p$ respectively. This set of fuzzy Boolean algebras is denoted as $B$, which is: $B = \{B_1, B_2, B_3,....B_p\}$.

The scalar multiplication among the fuzzy Boolean algebras is defined based on the membership values of the elements of the fuzzy subsets. For, any two fuzzy Boolean algebras $B_r, B_s \in B$, where, $1 \le r \le p, 1 \le s \le p$ and $r < s$, the scalar multiplication is defined as:

$$\mu_{B_r}(x_i) = \frac{r}{s}\mu_{B_s}(x_i), \forall x_i \in E,$$

Where, $i = 0,1,...n.$ again, $\mu_{B_r}(x_i)$ and $\mu_{B_s}(x_i)$ are the membership values of the $i^{th}$ element of the fuzzy subsets of $B_r$ and $B_s$ respectively.

## 3. Definition of Atom for a Fuzzy Boolean algebra

Every fuzzy Boolean algebra $F$ has a trivial subelement or fuzzy subset namely the set $\{\phi\}$, consisting the empty fuzzy subset alone; all other fuzzy subsets of $F$ are called non-trivial. Any fuzzy Boolean algebra $F$ has an improper fuzzy subset namely $'g'$ which is called the universal fuzzy subset or the top element; all other fuzzy subsets are called proper.

An atom of a fuzzy Boolean algebra $F$ is an element or fuzzy subset which does not have non-trivial proper fuzzy subsets. Alternatively, an atom of a fuzzy Boolean algebra $F$ is a fuzzy subset which cannot be expressed as the join (fuzzy union) of other non-trivial proper fuzzy subsets. This means that an element $v$ of a fuzzy Boolean algebra $F$ is an atom if $v \ne \phi$ and if there are only two elements $u$ such that $u \le v$, namely $\phi$ and $v$. Hence, the atoms of a fuzzy Boolean algebra are **1.1** the elements which cover the empty fuzzy subset. Atoms are also the immediate successor of the empty fuzzy subset, $\phi$.

### 3.1 Example

Let, $E = \{x_0, x_1, x_2\}$ be the universal set. Let, $M = \{0, h, 2h, 3h = 1\}$ is a set, where $h = \frac{1}{3}$.

Now, considering a mapping from $E$ to $\{0, h\}$, we get a fuzzy Boolean algebra $B_1$, written as follows:

$$\begin{aligned} B_1 = [0 &= \{(x_0,0),(x_1,0),(x_2,0)\}, \\ 1 &= \{(x_0,0),(x_1,0),(x_2,h)\}, \\ 4 &= \{(x_0,0),(x_1,h),(x_2,0)\}, \\ 5 &= \{(x_0,0),(x_1,h),(x_2,h)\}, \\ 16 &= \{(x_0,h),(x_1,0),(x_2,0)\}, \\ 17 &= \{(x_0,h),(x_1,0),(x_2,h)\}, \\ 20 &= \{(x_0,h),(x_1,h),(x_2,0)\}, \\ 21 &= \{(x_0,h),(x_1,h),(x_2,h)\}] \end{aligned}$$

Where, $'0'$ is the empty fuzzy subset or the bottom element and '21' is the universal fuzzy subset or the top element of the fuzzy Boolean algebra $B_1$.

The Hass diagram of the fuzzy Boolean algebra $B_1$ is shown in the Fig.1 bellow:

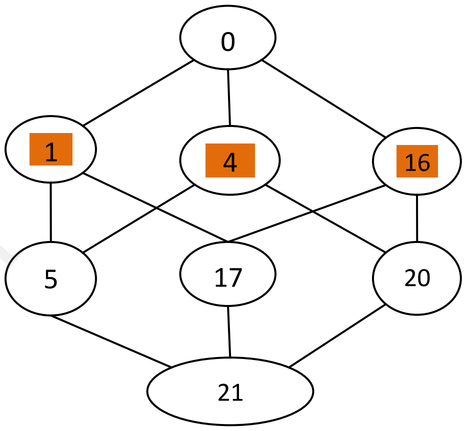


Fig.1 The Hass diagram of $B_1$

Here, the atoms are 1, 4 and 16 because only these fuzzy subsets cannot be expressed as the join of other non-trivial proper fuzzy subsets.

In the following, some characterizations of the atoms of the fuzzy Boolean algebras have been discussed:

### 3.2 Lemma

The number of atoms of a fuzzy Boolean algebra is always equal to the number of elements in the universal set.

**Proof**: For a finite set $E = \{x_0, x_1, x_2..., x_{n-1}\}$ with $n$ elements and a set $M$ of membership values, such that,

$$M = \left\{0, \frac{1}{p}, \frac{2}{p}, \frac{3}{p}, ..., \frac{p-1}{p}, \frac{p}{p} = 1\right\}$$
$$= \{0, h, 2h, 3h, ..., (p-1)h, ph\},$$

where, $h = \frac{1}{p}$ and $p$ is any positive integer.

Then for any mapping $E \to \{0, kh\}$, where $1 \le k \le p$, forms a fuzzy Boolean algebra.

So, by the type of mappings considered, we get exactly n-fuzzy subsets containing exactly one non-zero membership; which cannot be expressed as the join of non-trivial proper fuzzy subsets. Therefore, only those fuzzy subsets are the atoms for a fuzzy Boolean algebra.

Hence, the number of atoms of a fuzzy Boolean algebra is always equal to the number of elements in the universal set.

Again, for the mappings $E \to \{0, h\}$, $E \to \{0, 2h\}, ... E \to \{0, ph\}$, $p$-numbers of Fuzzy Boolean algebras can be obtained. Therefore, the total number of atoms that can be obtained from all the fuzzy Boolean algebras formed by the fuzzy subsets of a finite set is always equal to $n \times p$.

### 3.3 Lemma

In any fuzzy Boolean algebra, the atoms are the fuzzy subsets which contain exactly one non-zero membership element.

**Proof:** From the definition of fuzzy set union operation and from the fuzzy Boolean algebra that have been introduced in [11], it can be observed that only those fuzzy subsets which contain exactly one non-zero membership element cannot be expressed as the join of non trivial proper fuzzy subsets; all the other non trivial proper fuzzy subsets can be expressed as the join of other non trivial proper fuzzy subsets. Hence, these fuzzy subsets are the only atoms of a fuzzy Boolean algebra. **1.2**

### 3.4 Lemma

All the fuzzy Boolean algebras formed by the fuzzy subsets of a finite set have the same number of atoms. **1.3**

**Proof:** In the lemma 3.2, it is proved that the number of atoms of a fuzzy Boolean algebra is always equal to the number of elements in the universal set. Therefore, it is obvious that all the fuzzy Boolean algebra formed by the fuzzy subsets of the same universal set have the same number of atoms.

### 3.5 Lemma

The empty fuzzy subset of a fuzzy Boolean algebra is the infimum of the set of all atoms; on the other hand the universal fuzzy subset is the supremum of the set of all atoms.

**Proof**: Since, the atoms of a fuzzy Boolean algebra are the fuzzy subsets which contain exactly one non-zero membership, so, the set $A$ of all atoms of a fuzzy Boolean algebra $B$ is of the form as follows:

$$A = [A_0 = \{(x_0,0),(x_1,0),...,(x_{n-2},0),(x_{n-1},kh)\},$$
$$A_1 = \{(x_0,0),(x_1,0),...,(x_{n-2},kh),(x_{n-1},0)\},$$
$$..............................................................$$
$$..............................................................,$$
$$A_{n-2} = \{(x_0,0),(x_1,kh),...,(x_{n-2},0),(x_{n-1},0)\},$$
$$A_{n-1} = \{(x_0,kh),(x_1,0),...,(x_{n-2},0),(x_{n-1},0)\}].$$

Now, the supremum of all the atoms is the join (fuzzy union) of all the atoms, which is as follows:

$A_0 \vee A_1 \vee A_2 \vee .... \vee A_{n-1} =$
$\{(x_0,kh),(x_1,kh),...,(x_{n-2},kh),(x_{n-1},kh)\}$ =universal fuzzy subset.

Again, the infimum of all the atoms is the meet (fuzzy intersection) of all the atoms, which is as follows:

$A_0 \wedge A_1 \wedge A_2 \wedge .... \wedge A_{n-1} =$
$\{(x_0,0),(x_1,0),...,(x_{n-2},0),(x_{n-1},0)\}$ =empty fuzzy subset.

Therefore, is clear the empty fuzzy subset of a fuzzy Boolean algebra is the infimum of the set of all atoms; on the other hand the universal fuzzy subset is the supremum of the set of all atoms.

### 3.6 Lemma

A fuzzy Boolean algebra is isomorphic to the power set of atoms by the mapping which maps each element of the fuzzy Boolean algebra to the set of atoms it dominates.

**Proof**: The proof is illustrated in the following example:

### 3.7 Example

Considering the fuzzy Boolean algebra $B_1$ as shown in example 3.1, where $B_1 = \{0,1,4,5,16,17,20,21\}$,where $A = \{1,4,16\}$ is the set of atoms.

Now, defining a function $f$ which takes each element of $B_1$ to the set of atoms it dominates, we get:

$$f(0) = \{0\},\ f(1) = \{1\},\ f(4) = \{4\},\ f(16) = \{16\},$$
$$f(5) = \{1,4\},\ f(17) = \{1,16\},\ f(20) = \{4,16\},$$
$$f(21) = \{1,4,16\}$$

Again, the power set of $X$, $\rho(A) =$
$[\{0\},\{1\},\{4\},\{16\},\{1,4\},\{1,16\},\{4,16\},\{1,4,16\}]$

Hence, $f$ is one to one and onto. So it is isomorphic.

### 3.8 Lemma

The set of atoms of a fuzzy Boolean algebra has one-to-one correspondence to the set of atoms of another fuzzy Boolean algebra of the same universal set.

**Proof**: Considering a universal set $E=\{x_0,x_1,x_2...,x_{n-1}\}$ with $n$ elements. From the definition of atom of fuzzy Boolean algebra, the set of all atoms $A$ of a fuzzy Boolean algebra $B_1$ is of the following form:

$$A=[A_0=\{(x_0,0),(x_1,0),...,(x_{n-2},0),(x_{n-1},kh)\},$$
$$A_1=\{(x_0,0),(x_1,0),...,(x_{n-2},kh),(x_{n-1},0)\},$$
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$$A_{n-2}=\{(x_0,0),(x_1,kh),...,(x_{n-2},0),(x_{n-1},0)\},$$
$$A_{n-1}=\{(x_0,kh),(x_1,0),...,(x_{n-2},0),(x_{n-1},0)\}].$$

Similarly, with the same universal set we can get another the set of atoms $C$ of another fuzzy Boolean algebra $B_2$ of the form as follows:

$$C=[C_0=\{(x_0,0),(x_1,0),...,(x_{n-2},0),(x_{n-1},k_1h)\},$$
$$C_1=\{(x_0,0),(x_1,0),...,(x_{n-2},k_1h),(x_{n-1},0)\},$$ 1.4
.............................................................
.............................................................,
$$C_{n-2}=\{(x_0,0),(x_1,k_1h),...,(x_{n-2},0),(x_{n-1},0)\},$$
$$C_{n-1}=\{(x_0,k_1h),(x_1,0),...,(x_{n-2},0),(x_{n-1},0)\}].$$

Now, from the scalar multiplication that has been defined in [12], we can define a function $f$ from $A$ to $C$ such that, $f(A_0)=C_0$, $f(A_1)=C_1$,.....$f(A_{n-1})=C_{n-1}$. Hence, $f$ one to one and onto.

Therefore, it is clear that the set of atoms of a fuzzy Boolean algebra has one-to-one correspondence to the set of atoms of another fuzzy Boolean algebra of the same universal set.

### 3.9 Lemma

The number of elements of a fuzzy Boolean algebra= $2^{|atom|}$

**Proof:** Since, the number of elements in a fuzzy Boolean algebra= $2^{|E|}=2^{\text{no.of elements in the universal set}}$

Also, in case of fuzzy Boolean algebra, it is clear that:

Number of atoms=Number of elements in the universal set.

Hence, the number of elements of a fuzzy Boolean algebra= $2^{|atom|}$

## 4. Definition of Co-atom for Fuzzy Boolean algebra

Every fuzzy Boolean algebra $F$ has an improper element or fuzzy subset namely the universal fuzzy subset '$g$'; all other subsets are called proper.

The co-atoms of a fuzzy Boolean algebra are the fuzzy subsets which does not have any proper fuzzy superset. Alternatively, a co-atom of a fuzzy Boolean algebra is the fuzzy subset which cannot be expressed as the meet (intersection) of any proper fuzzy subsets. This implies that an element or fuzzy subsets $c$ of a fuzzy Boolean algebra $F$ is said to a co-atom if $c\neq g$, where '$g$' is the universal fuzzy subset and if there are only two elements or fuzzy subsets $d$ and $c$ such that $c\subseteq d$, namely '$g$' and '$c$'.

Hence co-atoms of a fuzzy Boolean algebra are the fuzzy subsets which are covered by the universal fuzzy subset '$g$' or which are the immediate predecessor of '$g$'.

The co-atoms of a fuzzy Boolean algebra is illustrated in the following numerical example

### 4.1 Example

Considering the fuzzy Boolean algebra $B_1$ once again as in example 3.1, where $B_1$ is written as follows:

$$B_1=[0=\{(x_0,0),(x_1,0),(x_2,0)\},$$
$$1=\{(x_0,0),(x_1,0),(x_2,h)\},$$
$$4=\{(x_0,0),(x_1,h),(x_2,0)\},$$
$$5=\{(x_0,0),(x_1,h),(x_2,h)\},$$
$$16=\{(x_0,h),(x_1,0),(x_2,0)\},$$
$$17=\{(x_0,h),(x_1,0),(x_2,h)\},$$
$$20=\{(x_0,h),(x_1,h),(x_2,0)\},$$
$$21=\{(x_0,h),(x_1,h),(x_2,h)\}]$$

The Hass diagram of $B_1$ is shown in the Fig. 2. bellow:

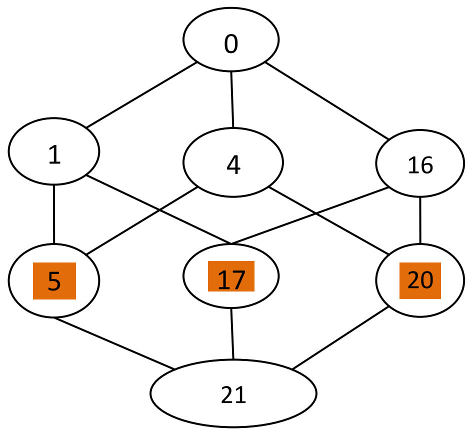


Fig.2 The Hass diagram of $B_1$

Here, from the definition, the co-atoms of $B_1$ are 5, 17 and 20.

Some characteristics of the co-atoms of fuzzy Boolean algebra are discussed as follows:

## 4.2 Lemma

In a fuzzy Boolean algebra, the co-atoms are the fuzzy subsets which contain exactly one zero membership.

**Proof**: From the definition of fuzzy set intersection operation and from the fuzzy Boolean algebra have been introduced it can be observed that all the proper fuzzy subsets except those which contains exactly one zero membership can be expressed as the meet of proper fuzzy subsets. This implies that only these elements are covered by the element universal fuzzy subset $'g'$. Hence, only these elements or fuzzy subsets are the only co-atoms of a fuzzy Boolean algebra.

## 4.3 Lemma

The number of co-atoms of a fuzzy Boolean algebra is always equal to the numbers of elements in the universal set.

**Proof**: For a finite set $E=\{x_0,x_1,x_2...,x_{n-1}\}$ with $n$ elements with the set $M$ of membership values, such that,

$$M=\left\{0,\frac{1}{p},\frac{2}{p},\frac{3}{p},...,\frac{p-1}{p},\frac{p}{p}=1\right\}$$
$$=\{0,h,2h,3h,...,(p-1)h,ph\},$$

where, $h=\dfrac{1}{p}$ and $p$ is any positive integer.

Since, for any mapping $E\to\{0,kh\}$, where $1\le k\le p$, forms a fuzzy Boolean algebra. So, there are exactly n-fuzzy subsets containing exactly one zero membership; which cannot be expressed as the meet of proper fuzzy subsets. Therefore, only these fuzzy subsets are the co-atoms of a fuzzy Boolean algebra. Hence, the number of co-atom of a fuzzy Boolean algebra is always equal to the number of elements in the universal set.

Therefore, the number of atoms and co-atoms of any fuzzy Boolean algebra is the same.

## 4.4 Lemma

The co-atoms and atoms of a fuzzy Boolean algebra are the complement of each other.

**Proof:** Since, the atoms of a fuzzy Boolean algebra are the fuzzy subsets which contain exactly one non-zero membership, so, the set of all atoms $A$ of a fuzzy Boolean algebra $F$ is of the form as follows:

$$A=[A_0=\{(x_0,0),(x_1,0),...,(x_{n-2},0),(x_{n-1},kh)\},$$
$$A_1=\{(x_0,0),(x_1,0),...,(x_{n-2},kh),(x_{n-1},0)\},$$
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.............................................................,
$$A_{n-2}=\{(x_0,0),(x_1,kh),...,(x_{n-2},0),(x_{n-1},0)\},$$
$$A_{n-1}=\{(x_0,kh),(x_1,0),...,(x_{n-2},0),(x_{n-1},0)\}].$$

Again, since, the co-atoms of a fuzzy Boolean algebra are the fuzzy subsets which contain exactly one zero membership, so, the set of all co-atoms $C$ of the fuzzy Boolean algebra $F$ is of the form as follows:

$$C=[C_0=\{(x_0,0),(x_1,0),...,(x_{n-2},0),(x_{n-1},kh)\},$$
$$C_1=\{(x_0,0),(x_1,0),...,(x_{n-2},kh),(x_{n-1},0)\},$$
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.............................................................,
$$C_{n-2}=\{(x_0,0),(x_1,kh),...,(x_{n-2},0),(x_{n-1},0)\},$$
$$C_{n-1}=\{(x_0,kh),(x_1,0),...,(x_{n-2},0),(x_{n-1},0)\}].$$

From the definition of complementation it follows that: $(A_0)'=C_0,(A_1)'=C_1,....(A_{n-1})'=C_{n-1}$.

Therefore, co-atoms and atoms of a fuzzy Boolean algebra are the complement of each other.

## 4.5 Lemma

The empty fuzzy subset of a fuzzy Boolean algebra is the infimum of the set of all co-atoms; on the other hand the universal fuzzy subset is the supremum of the set of all co-atoms.

**Proof:** Since, the co-atoms of a fuzzy Boolean algebra are the fuzzy subsets which contain exactly one zero membership, so, the set of all co-atoms $C$ of a fuzzy Boolean algebra $B$ is of the form as follows:

$$C=[C_0=\{(x_0,0),(x_1,0),...,(x_{n-2},0),(x_{n-1},kh)\},$$
$$C_1=\{(x_0,0),(x_1,0),...,(x_{n-2},kh),(x_{n-1},0)\},$$
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.............................................................,
$$C_{n-2}=\{(x_0,0),(x_1,kh),...,(x_{n-2},0),(x_{n-1},0)\},$$
$$C_{n-1}=\{(x_0,kh),(x_1,0),...,(x_{n-2},0),(x_{n-1},0)\}].$$

Now, the supremum of all the co-atoms is the join (fuzzy union) of all the co-atoms, which is as follows:

$C_0\vee C_1\vee C_2\vee ....\vee C_{n-1}=$
$\{(x_0,kh),(x_1,kh),...,(x_{n-2},kh),(x_{n-1},kh)\}$=universal fuzzy subset.

Again, the infimum of all the atoms is the meet (fuzzy intersection) of all the atoms, which is as follows:

$C_0 \wedge C_1 \wedge C_2 \wedge .... \wedge C_{n-1} =$

$\{(x_0,0),(x_1,0),...,(x_{n-2},0),(x_{n-1},0)\}$ =empty fuzzy subset.

Therefore, it is clear that empty fuzzy subset of a fuzzy Boolean algebra is the infimum of the set of all co-atoms; on the other hand the universal fuzzy subset is the supremum of the set of all co-atoms.

Again it is proved that the empty fuzzy subset of a fuzzy Boolean algebra is also the infimum of the set of all atoms and the universal fuzzy subset is the supremum of the set of all atoms. So, this is a correspondence between the set of all atoms and the set of all co-atoms of a fuzzy Boolean algebra.

### 4.6 Lemma:

The set of co-atoms of a fuzzy Boolean algebra has one-to-one correspondence with the set of co-atoms of another fuzzy Boolean algebra of the same finite set.

**Proof**: The proof is obvious from the definition of scalar multiplication.

### 4.7 Lemma:

A fuzzy Boolean algebra is isomorphic to the power set of co-atoms by the mapping which maps each element of the fuzzy Boolean algebra to the set of co-atoms it precedes.

**Proof**: Considering the fuzzy Boolean algebra $B$ as shown in example1, where $B = \{0,1,4,5,16,17,20,21\}$ ,where $A = \{5,17,20\}$ is the set of co-atoms.

Now, defining a function $f$ which takes each element of $B$ to the set of co-atoms it precedes, we get:

$f(0) = \{0\}$, $f(1) = \{5,17\}$, $f(4) = \{5,20\}$,

$f(16) = \{17,20\}$, $f(5) = \{5\}$, $f(17) = \{17\}$,

$f(20) = \{20\}$, $f(21) = \{5,17,20\}$

Again, the power set of $X$ , $\rho(A) =$

$[\{0\},\{5\},\{17\},\{20\},\{5,17\},\{5,20\},\{17,20\},\{5,17,20\}]$

Hence, $f$ is one to one and onto. So, $f$ is isomorphic.

## 5. CONCLUSIONS

This article introduces the concept of atoms and co-atoms of the fuzzy Boolean algebras formed by the fuzzy subsets of a finite set that have been introduced in [11]. Further, some properties of the atoms and co-atoms are discussed which can be a foundation. But there is a lot of potential growth in this direction.

## 6. REFERENCES

[1] Byliński, C. (1990). Functions from a set to a set. *Formalized Mathematics*, *1*(1), 153-164.

[2] Dubois, Didier J. *Fuzzy sets and systems: theory and applications*. Vol. 144. Access Online via Elsevier, 1980.

[3] Frink, O. (1954). Ideals in partially ordered sets. The American Mathematical Monthly, 61(4), 223-234.

[4] Givant, S., Halmos, P.: Introduction to Boolean Algebras. Undergrad. Texts Math. Springer, New York (2009).

[5] Klir, George J., and Bo Yuan. *Fuzzy sets and fuzzy logic*. New Jersey: Prentice Hall, 1995.

[6] Monk, J. D., & Bonnet, R. (1989). *Handbook of Boolean algebras* (Vol. 3). North Holland.

[7] Rosen, K. H., & Krithivasan, K. (1999). *Discrete mathematics and its applications* (Vol. 6). New York: McGraw-Hill.

[8] Sikorski, R., Sikorski, R., Sikorski, R., Sikorski, R., & Mathematician, P. (1969). *Boolean algebras* (Vol. 2). New York: Springer.

[9] Squires, R. J. An introduction to Boolean algebra. Journal of the Institute of Actuaries Students' Society 17 (1964):314.

[10] Stone, M. H. (1936). The theory of representation for Boolean algebras. *Transactions of the American Mathematical Society*, *40*(1), 37-111.

[11] Sisir Kumar Rajbongshi, and Dwiraj Talukdar, "Some Aspects of fuzzy Boolean algebras formed by fuzzy subsets", IJARCSSE, 3.7 (2013): 1-8.
Dwiraj Talukdar, and Sisir Kumar Rajbongshi. "An
[12] Introduction to a Family of Fuzzy Subsets forming Boolean algebra." *International Journal of Computer Applications* 68.24 (2013): 1-6.