Some Properties of Semigraph and its Associated Graphs

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***Abstract*— In this paper, some properties of degrees of vertices of a semigraph have been discussed. Four types of graphs associated to a given semigraph have been defined, and size of each graph has also been discussed.**

***Keywords*— Semigraph, Adjacency and Degrees in Semigraph, Associated graphs.**

I. INTRODUCTION

Semigraph is a generalization of the concept of graph [3,5]. In a semigraph the edges are assumed to have many parts, and vertices are classified according to their positions [1,2]. In this paper, the basic characteristics of all types of vertices have been discussed in detail. In particular, four types of degrees have been defined for each vertex and relationship among them have been studied in detail. It has also been discussed four types of associated graphs and size of each of them corresponding to the given semigraph.

II. SEMIGRAPH

A. Definition

A semigraph S is a pair (V, X) where V is a nonempty set whose elements are called vertices of S and X is a set of ordered n-tuples $n \geq 2$, called edges of S satisfying the following conditions [4]:

i. The components of an edge E in X are distinct vertices from V.
ii. Any two edges have at most one vertex in common.
iii. Two edges $E_1 = (u_1, u_2, ...u_m)$ and $E_2 = (v_1, v_2, ...v_n)$ are said to be equal iff
   a. $m = n$ and
   b. Either $u_i = v_i$ or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

The vertices in a semigraph are divided into three types namely end vertices, middle vertices and middle-end vertices, depending upon their positions in an edge. The end vertices are represented by thick dots, middle vertices are represented by small circles, a small tangent is drawn at the small circles to represent middle-end vertices.

B. Example

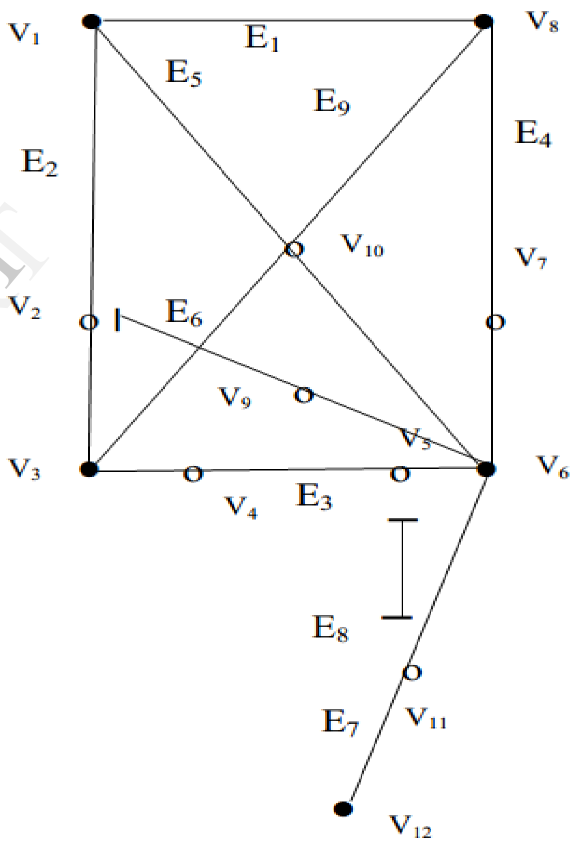

Fig.1 A Semigraph

In the semigraph S, given in fig.1 $V = \{v_1, v_2, ...v_{10}, v_{11}, v_{12}\}$ and $X = \{(v_1, v_8), (v_1, v_2, v_3), (v_3, v_4, v_5, v_6), (v_6, v_7, v_8), (v_6, v_{10}, v_1), (v_6, v_9, v_2), (v_6, v_{11}, v_{12}), (v_5, v_{11}), (v_3, v_{10}, v_8)\}$.

The vertices v_1, v_3, v_6, v_8 and v_{12} are end vertices, v_4, v_7, v_9, v_{10} are middle vertices and v_2, v_5, v_{11} are middle-end vertices.

The order and size of a semigraph S is respectively denoted by the numbers $|V|$ and $|E|$. The number of components in an edge E is denoted by $|E|$, we also write $V = V_1 \cup V_2 \cup V_3$, where V_1, V_2 and V_3 respectively represent the set of end vertices, middle vertices and middle-end vertices.

III. Adjacent Vertices

There are four types of adjacency is defined between any two vertices in a semigraph S.

A. *Definition*

Two vertices u and v in a semigraph S are said to be

i. adjacent if they belong to the same edge.

ii. consecutively adjacent if they are adjacent and consecutive in order as well.

iii. e-adjacent if they are end vertices of an edge.

iv. 1e-adjacent if both the vertices u and v belong to the same edge and atleast one of them is an end vertex of that edge.

As an example, consider the semigraph S given in fig 1. The vertices v_1, v_2, v_3 are adjacent, v_2, v_3 are consecutively adjacent, v_1, v_3 are e-adjacent, and v_1, v_2 are 1e-adjacent.

Also from the above example, the following can be observed:

B. *Observations*

i. The number of adjacent vertices to a vertex v in an edge E is $|E| - 1$.

ii. The adjacency of two vertices does not imply the consecutive adjacent, but the converse is always true.

iii. The number of e-adjacent vertices to an end vertex v in a semigraph is equal to the number of edges for which v acts as an end vertex.

iv. The number of e-adjacent vertices to a middle vertex is always zero.

v. Let v be an end vertex, then the number of vertices 1e-adjacent to v is

$$\sum_{i=1}^{n} |E_i| - 2n$$, where v is an end vertex to the edges $E_1, E_2, \ldots E_n$.

IV. Various Degrees of Vertices

A. *Definition*

For a vertex v in a semigraph $S = (V, X)$, the various types of degrees are defined as follows:

i. Degree of a vertex v, is the number of edges having v as an end vertex. It is denoted as $\deg(v)$.

ii. Edge degree of a vertex $d_e(v)$, is the number of edges containing v.

iii. Adjacent degree of a vertex $d_a(v)$, is the number of vertices adjacent to v.

iv. Consecutive adjacent degree of a vertex $d_{ca}(v)$, is the number of vertices consecutively adjacent to v.

The following table gives the degree, edge degree, adjacent degree and consecutively adjacent degree of all the vertices of the semigraph given in fig.1

TABLE I

End Vertices	v_1	v_3	v_6	v_8	v_{12}
$\deg(v)$	3	3	5	3	1
$d_e(v)$	3	3	5	3	1
$d_a(v)$	5	7	11	5	2
$d_{ca}(v)$	2	3	5	3	1
Middle Vertices	v_4	v_7	v_9	v_{10}	-
$\deg(v)$	0	0	0	0	-
$d_e(v)$	1	1	1	2	-
$d_a(v)$	3	2	2	4	-
$d_{ca}(v)$	2	2	2	4	-
Middle-End Vertices	v_2	v_5	v_{11}	-	-
$\deg(v)$	1	1	1	-	-
$d_e(v)$	2	2	2	-	-
$d_a(v)$	4	4	3	-	-
$d_{ca}(v)$	3	3	3	-	-

The following theorems give the properties of various degrees of vertices and relationship between one another.

B. Theorem.

For any semigraph $S=(V,X)$, and $v\in V$

i. $\sum \deg(v)=2|X|$.

ii. $\sum d_e(v)=\sum \deg(v)+|X_2|+|X_3|=2|X|+|X_2|+|X_3|$

Proof:

i. It is simple observation that if $v\in V_2$, then $\deg(v)=0$. Suppose if $v\in V_1$, and $\deg(v)=m$, then there are m edges in X, such that v is an end vertex for all such m edges. Each edge also contains another end vertex other than v. Hence each edge counted twice respectively for each end vertex. The same is true for every vertex $v\in V_3$. This proves (i).

ii. Note that each edge is counted twice while calculating edge degree of end vertex, and middle-end vertex of that edge. Further each edge is again counted once for each middle vertex and middle-end vertex. In particular, if v is a middle-end vertex, such that it is a middle vertex for E_1, and an end vertex for E_2, both E_1 and E_2 are counted once each corresponding to the vertex v. The count of E_1 is included in the sum $|V_3|$ where as the count of E_2 is included in the sum $\sum \deg(v)=2|X|$. This proves (ii).

C. Theorem

If $S=(V,X)$ is a semigraph, and $v\in V_1$, then $\deg(v)=d_e(v)=d_{ca}(v)$.

Proof:

Since, the number of edges for which a vertex $v\in V_1$ acts as an end vertex is same as the number of edges containing it. Hence $\deg(v)=d_e(v)$. Also if v is an end vertex for n edges $E_1,E_2,\ldots..E_n$, then there is exactly one vertex v_i in each E_i consecutively adjacent to v. Hence the number of vertices consecutively adjacent to v is same as the number of edges for which v acts as an end vertex. Hence $\deg(v)=d_e(v)=d_{ca}(v)$.

D. Theorem

If $S=(V,X)$ is a semigraph, and $v\in V_2$, then

i. $\deg(v)=0$

ii. $d_e(v)\geq 1$

iii. $d_a(v)=|E|-1$, provided $v\in E$, an edge in X

iv. $\sum d_{ca}(v)=2n, if\ |X_2|=n$

v. $d_{ca}(v)=2d_e(v)$

Proof:

The proof of (i) is obvious, because $\deg(v)$ is defined as the number of edges containing v as end vertex. Since $v\in V_2$, no such edge exists. Hence $\deg(v)=0$.

It is also easy to see that middle vertices in a semigraph S belong to atleast one edge in X. Hence $d_e(v)\geq 1$. This proves (ii).

Let $v\in E=(u_1,u_2,\ldots.,v,\ldots u_n)$. Then the vertices $u_1,u_2,\ldots u_n$ are all adjacent vertices to v corresponding to the edge E. Therefore $d_a(v)=n=|E|-1$. This proves (iii).

Let $v\in V_2$ .Then $v\in E=(u_1,u_2,\ldots.,u_{i-1},v,u_i,\ldots u_n)$ for some E in X. Note that u_{i-1} and u_i are consecutively adjacent to v. Since $v\in V_2$, existence of two vertices such as u_{i-1} and u_i are must to v. Therefore $d_{ca}(v)=2$, corresponding to the edge E containing v. Moreover, if there are n middle vertices in S, then the sum of consecutively adjacent degree of all n vertices is $2n$. This proves (iv).

If $v\in V_2$, and E contains v, then v has two consecutively adjacent vertices in E. This is true for every edge containing v. In particular, if there are n edges containing v, then $d_{ca}(v)=2n$, and $d_e(v)=n$. Hence $d_{ca}(v)=2d_e(v)$. This proves (v).

Hence the theorem.

E. Theorem

If $S=(V,X)$ be a semigraph, and $v\in V_3$, then

i. $d_e(v)>\deg(v)$

ii. $d_{ca}(v) = 2n(E_i) + n(E_j)$, where $n(E_i)$ denotes number of edges in S at which v is a middle vertex, and $n(E_j)$ denotes the number of edges in S at which v is an end vertex in S.

Proof:

The proof (i) follows from the fact that $\deg(v)$ is calculated by considering only the edges for which v is an end vertex, where as the $d_e(v)$ is calculated by considering all the edges containing v. Therefore $d_e(v) > \deg(v)$.

Let $v \in V_3$. Then there are edges $E_i, i = 1, 2, ..., n$ and $E_j, j = 1, 2, ..., m$ such that v is a middle vertex for all E_i, and end vertex for all E_j. For every E_i, $d_{ca}(v) = 2$ and for every $E_j, d_{ca}(v) = 1$. Therefore if $n(E_i) = n$, and $n(E_j) = m$, then $d_{ca}(v) = 2n(E_i) + n(E_j)$. Hence the theorem.

F. Theorem

If $S = (V, X)$ is a semigraph, and $v \in V$, then

$$\sum_{v \in V} d_a(v) = \sum_{E_i} |E_i|(|E_i| - 1).$$

Proof:

Let $S = (V, X)$ is a semigraph, and $v \in V$. Also let $v \in E_i$, for some E_i in X. If $|E_i| = n$, then for every $v \in E_i$, $d_a(v) = n - 1$. Hence for all $v \in E_i$, $\sum_{v \in E_i} d_a(v) = n(n-1)$. On generalizing the above fact, we can write $\sum_{v \in V} d_a(v) = \sum_{E_i} |E_i|(|E_i| - 1)$.

This proves the theorem.

V. ASSOCIATED GRAPHS

Let $S = (V, X)$ be a semigraph. We define four new graphs associated to the given semigraph each having same vertex set.

A. Definition

i. End Vertex Graph S_e : Two vertices in S_e are adjacent if they are the end vertices of an edge in S.

ii. Adjacency Graph S_a: Two vertices in S_a are adjacent if they are adjacent in S.

iii. Consecutive adjacency Graph S_{ca}: Two vertices in S_{ca} are adjacent if they are the consecutively adjacent in S.

iv. One end vertex Graph S_{1e}: Two vertices in S_{1e} are adjacent if atleast one of them is an end vertex in S of an edge containing the two vertices.

B. Example

The above four types of associated graphs for the semigraph given fig.1 are shown below.

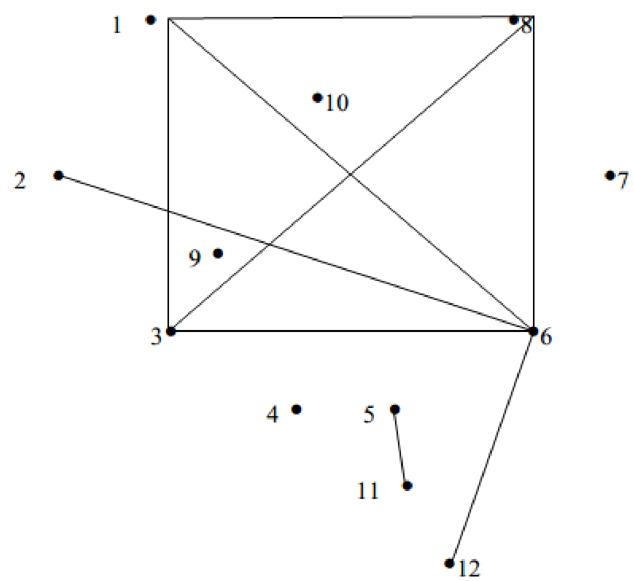

Fig 2. S_e

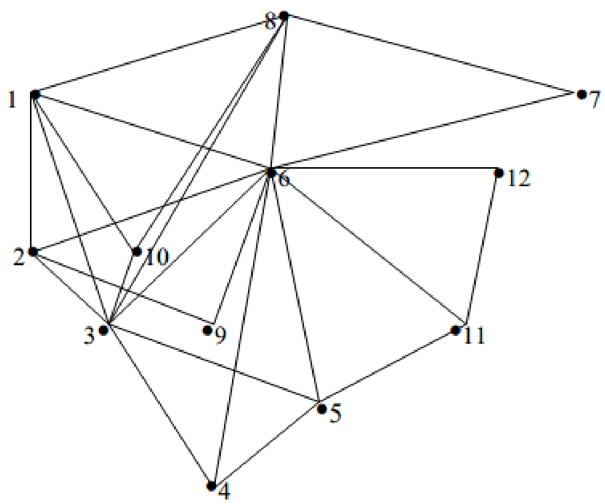

Fig. 3 S_a

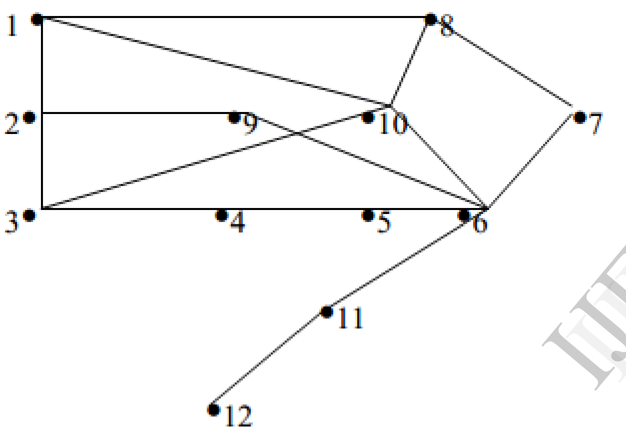

Fig.4 S_{ca}

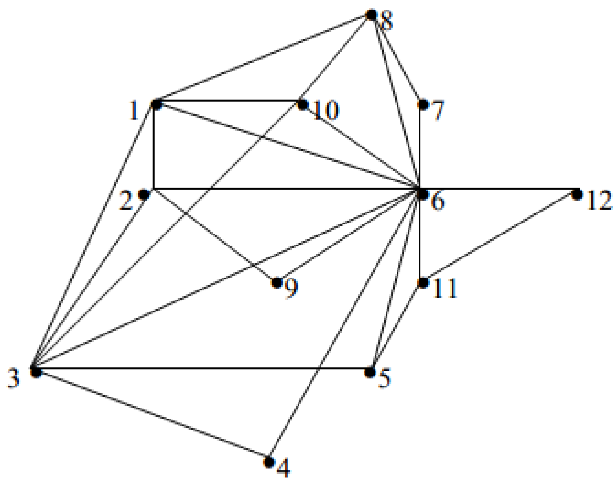

Fig. 5 S_{1e}

C. Theorem

End vertex graph associated with a given semigraph is connected if the semigraph has no middle and middle-end vertices.

Proof:

In an end vertex graph, only the end vertices of an edge are adjacent. Also, if there are two middle-end vertices joined by an edge, then they produce a component in the end vertex graph. Moreover, the middle vertex is not adjacent to any other vertex in the end vertex graph. This leads to fact that the middle vertices in the semigraph are isolated vertices in the end vertex graph. Hence the end vertex graph of the corresponding semigraph is connected if it has no middle vertices and middle-end vertices.

D. Note

i. Every graph can be considered as a semigraph with no middle and middle end vertices. In this case the end vertex graph is the same graph.

ii. Let S be a semigraph containing end vertices, and middle-end vertices but not middle vertices. Then the end vertex graph G associated with S has pendent vertices, components but not isolated vertices.

E. Theorem

The size of S_a is $\sum_i |E_i| C_2$, where E_i is an edge in S.

Proof:

In a semigraph S, two vertices are adjacent if they belong to same edge. Therefore all vertices belonging to an edge in S, become adjacent to each other in S_a. i.e., the vertices of each edge E_i in S form a clique in S_a consisting $|E_i| C_2$ edges. Hence the total number of edges in S_a is $\sum_i |E_i| C_2$.

As in the proof of above theorem we can also prove the following theorem.

F. Theorem

The size of S_{ca} is $\sum_i \left(|E_i| - 1\right)$, where E_i is an edge in S.

G. Theorem

The size of S_e is same as the size of S.

H. Theorem

The size of S_{1e} is $\sum_i \left(|E_i| C_2 - |F_i| C_2\right)$, where E_i is an edge in S, and F_i is a set consisting the middle, and middle-end vertices in .

VI. CONCLUSION

The concept of adjacency between vertices in a semigraph plays a vital role in the domination theory. There are four types of adjacency have been discussed as far as a semigraph is concerned. Hence, it may be interesting to study the variations of domination properties based on the adjacency under consideration.

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