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Spherically Symmetric Self-Gravitating Fluid With Specified Equation Of State

by

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ABSTRACT

In this paper some exact, static spherically symmetric solution of Einstein's field equations for the perfect fluid with equation of state p=aρ, where a∈[0,1] has been obtained taking suitable choice of g_{11} or g_{44} (e.g. $e^{-\lambda} = k$ or $e^{\upsilon} = Ar^{n}$). Many previously known solutions are contained here in as a particular case. Various physical and geometrical properties have been also studied.

Key words: Exact solution, Perfect fluid, Equation of state. Homogenous, Homaloidal.

1. INTRODUCTION

Perfect fluid spheres with homogeneous density and isotopic pressure in general relativity were firstly considered by Schwarzschild [10] and the solutions of relativistic field equations were obtained. Tolman [16] developed a mathematical method for solving Einstein's field equations applied to a static fluid sphere in such a manner as to provide explicit solutions in terms of known functions. A number of new solutions were thus obtained and the properties of three of them were examined in detail.

No stationary inhomogeneous solutions to Einstein's equation for an irrotational perfect fluid have featured equations of state. p=ρ (Letelier [14], Letelier and Tabensky [15] and Singh and Yadav [23]). Solutions to Einsteins equation with a simple equation of state have been found in various cases, e.g. for ρ+3p=constant (Whittaker [7]) for ρ=3p (Klein [12], Singh and Abdussattar [11], Feinstein and Senovilla [1], Kramer [2]); for p=ρ+constant (Buchdahl and Land [6], Alluntt [9]) and for $\rho = (1+a)\sqrt{p} - ap$ (Buchdahl [4]). But if one takes, e.g. polytropic fluid sphere $\rho = ap^{1+\frac{1}{n}}$ (Klein [12], Tooper [18], Buchdahl [5]) or a mixture of ideal gas and radiation (Suhonen [3]), one soon has to use numerical methods. Yadav and Saini [20] have also studied the static fluid sphere with equation of state p=ρ (i.e. stiff matter). Davidson [25] has presented a solution that provides a non stationary analog to the static case when $p = \frac{1}{3}\rho$.

In the present paper, we have obtained some exact, static spherically symmetric solutions of Einstein's field equations for the perfect fluid with equation of state p=aρ, where a∈ [0,1]. We have also taken $e^{\lambda} = k_1$ in one case while $e^{\upsilon} = Ar^n$ in second case. For different values of a and n we get many previously known solutions. To overcome the difficulty of infinite density at the centre, it is assumed that distribution has a core of radius r_o and constant density ρ_o which is surrounded by the fluid with the specified equation of state.

2. THE FIELD EQUATIONS AND THEIR SOLUTIONS

We take the line element in the form

(2.1) $$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\upsilon}dt^2$$

where λ and ν are functions of r only.

The field equations

(2.2) $$R^i_j - \frac{1}{2}R\delta^i_j = -8\pi T^i_j$$

for (2.1) are [1]

(2.3) $$-8\pi T^1_1 = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}$$

(2.4) $$-8\pi T^2_2 = -8\pi T^3_3 = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu^1 - \lambda'}{2r}\right)$$

(2.5) $$-8\pi T^4_4 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

where a prime denotes differentiation with respect to r.

The energy momentum tensor for perfect fluid is given by

(2.6) $$T^i_j = (\rho + b)u^i u_j - \delta^i_j p.$$

We choose the equation of state as

(2.7) ρ=ap

where a is positive constant $a \in [0,1]$

In this case we find that

$$T^i_j = 0 (i \neq j)$$

We use commoving co-ordinate so that

$$u^1 = u^2 = u^3 = 0 \text{ and } u^4 = e^{-\nu/2}$$

The non-vanishing components of the energy momentum tensor are

$$T^1_1 = T^2_2 = T^3_3 = -p \text{ and } T^4_4 = \rho$$

We can then write

(2.8) $$8\pi p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2},$$

(2.9) $$8\pi p = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu^1 - \lambda'}{2r}\right),$$

(2.10) $$8\pi\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

Using equations (2.7), (2.8) and (2.10) we get

(2.11) $$ae^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{a}{r^2} = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}.$$

Case I

We choose $e^{\lambda} = k_1$ (a constant) which reduces (2.11) to the form

(2.12) $$\nu' + \frac{1}{r}\left[(1 - k_1)\left(1 + \frac{1}{a}\right)\right] = 0$$

Integrating w.r.t. r, we get

(2.13) $$e^{\upsilon} = k_2 r^{(k_1 - 1)\left(1 + \frac{1}{a}\right)}$$

where k_2 is a constant. Now (2.8) and (2.9) lead to $k_1=2$, so that

(2.14) $e^{\upsilon} = k_2 r^{\left(1+\frac{1}{a}\right)}$

Hence the metric (2.1) can be cast into the form

(2.15) $ds^2 = -2dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + k_2 r^{\left(1+\frac{1}{a}\right)} dt^2$

Absorbing the constant k_2 is the co-ordinate differential dt the metric (2.15) is reduced to the form.

(2.16) $ds^2 = -2dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + r^{\left(1+\frac{1}{a}\right)} dt^2$

The non zero components of Reimann-Christoffel curvature tensor R_{hijk} for the metric (2.16) are

(2.17) $R_{2424}\sin^2\theta = \frac{R_{2323}}{2}\left(1+\frac{1}{a}\right) r^{\left(\frac{1}{a}-1\right)} = R_{3434} = \left(-\frac{1}{4}\right)\left(1+\frac{1}{a}\right) r^{\left(1+\frac{1}{a}\right)} \sin^2\theta$

Choosing the orthonormal tetrad $\bar{\lambda}_j^i$ as

(2.18) $$\begin{cases} \bar{\lambda}_1^i = \left(\frac{1}{\sqrt{2}}, \quad 0, \quad 0, \quad 0\right) \\ \bar{\lambda}_2^i = \left(0, \quad \frac{1}{r}, \quad 0, \quad 0\right) \\ \bar{\lambda}_3^i = \left(0, \quad 0, \quad \frac{1}{r\sin\theta}, \quad 0\right) \\ \bar{\lambda}_4^i = \left(0, \quad 0, \quad 0, \quad \frac{1}{r^{\frac{a+1}{2a}}}\right) \end{cases}$$

The physical components $R_{(abcd)}$ of the curvature tensor defined by

$R_{(abcd)} = \bar{\lambda}_{(a)}^h \bar{\lambda}_{(b)}^i \bar{\lambda}_{(c)}^j \bar{\lambda}_{(d)}^k R_{hijk}$

are

(2.19) $R_{2424} = \left(\frac{a+1}{2}\right) R_{2323} = R_{3434} = -\frac{(a+1)}{4ar^2}.$

Since a is finite +ve constant, we see that

$R_{(abcd)} \to 0 \quad \text{as} \quad r \to \infty.$

Hence it follows that the space time is asymptotically homaloidal. For the metric (2.16) the fluid velocity u^i is given by

(2.20) $u^1 = u^2 = u^3 = 0 = u_1 = u_2 = u_3 \quad \text{and} \quad u_4 = \frac{1}{r^{\left(\frac{a+1}{2a}\right)}}, u^4 = r^{\frac{a+1}{2a}}.$

The scalar of expansion $\Theta = u^i_{;i}$ is identically zero. The non vanishing components of the tensor of rotation ω_{ij} is defined by

(2.21) $\omega_{ij} = u_{i;j} - u_{j;i}$

are

(2.22) $\omega_{14} = -\omega_{41} = -\left(\frac{a+1}{2a}\right) r^{\left(\frac{1-a}{2a}\right)}$

The components of the shear tensor σ_{ij} defined by

(2.23) $\sigma_{ij} = \frac{1}{2}\left(u_{i;j} + u_{j;1}\right) - \frac{1}{3}\Theta h_{ij}$

with the projection tensor

(2.24) $h_{ij} = g_{ij} - u_i u_j$

are

(2.25) $\sigma_{14} = \sigma_{41} = \frac{1}{2}.\left(\frac{a+1}{2a}\right) r^{\left(\frac{1-a}{2a}\right)}.$

while other components are zero.

For the particular values of constant a, several previously known solutions are contained here in. When a=1, results of this case reduce to that of Singh and Yadav [23]. Also in this case the relative mass m of a particle in the gravitational field of (2.16) is related to its proper mass m_o (Narlikar [24]) through

(2.26) $\frac{m}{m_o} = \frac{k^2}{r^2}$

k being a constant. As the particle moves towards the origin, m increases and $r \to \infty$, $m \to 0$ i.e. the relative mass goes on decreasing continuously.

The case when a=3 gives the distribution of disordered radiation already obtained by Singh and Abdussattar [11].

Case II

From (2.11) we see that if ν is known, λ can be obtained. So we choose

(2.27) $e^{\nu} = Ar^n$

where A is constant

Use of (2.27) reduces the equation (2.11) to the form

(2.28) $e^{-\lambda}(an + a - \lambda' r + 1) = a + 1.$

We put $y = e^{-\lambda}$ so that equation (2.28) is transformed to

(2.29) $\frac{dy}{dr} + (an + a + 1)\frac{y}{r} = \frac{a+1}{r}$

which is a linear differential equations whose solution is

(2.30) $y = \frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}}$

where E is integration constant.

Therefore we get

(2.31) $e^{-\lambda} = \frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}}$.

Consequently the metric (2.1) can be put into that form

(2.32) $ds^2 = Ar^n dt^2 - \left(\frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}}\right)^{-1} dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$

Absorbing the constant A in co-ordinate differential dt, the metric (2.32) goes to the form

(2.33) $ds^2 = r^n dt^2 - \left(\frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$

The non vanishing components of Reimann-Christoffel curvature tensor R_{hijk} for the metric (2.33) are

(2.34)
$$\begin{cases} R_{1212} = \dfrac{E(an+a+1)}{2\left(\dfrac{a+1}{an+a+1} + \dfrac{E}{r^{an+a+1}}\right).r^{an+a+1}}, \\ R_{1313} = \dfrac{E(an+a+1)\sin^2\theta}{2r^{an+a+1}\left(\dfrac{a+1}{an+a+1} + \dfrac{E}{r^{an+a+1}}\right).}, \\ R_{1414} = r^{n-2}\left[\dfrac{2n-n^2}{4}\right] + \dfrac{nE(an+a+1)}{4r^{\{n(a-1)+(a+3)\}}\left(\dfrac{a+1}{an+a+1} + \dfrac{E}{r^{an+a+1}}\right)}, \\ R_{2323} = -r^2\sin^2\theta\left(\dfrac{a+1}{an+a+1} + \dfrac{E}{r^{an+a+1}}\right), \\ R_{2424} = -\dfrac{nr^n}{2}\left(\dfrac{a+1}{an+a+1} + \dfrac{E}{r^{an+a+1}}\right), \\ R_{3434} = -\left(\dfrac{nr^n\sin^2\theta}{2}\right)\left(\dfrac{a+1}{an+a+1} + \dfrac{E}{r^{an+a+1}}\right). \end{cases}$$

Choosing the orthonormal tetrad $\bar{\lambda}_j^i$ as

$$(2.35)\begin{cases} \bar{\lambda}_1^i = \left(\left(\frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}} \right)^{1/2}, \quad 0, \quad 0, \quad 0 \right) \\ \bar{\lambda}_2^i = \left(0, \quad \frac{1}{r}, \quad 0, \quad 0 \right) \\ \bar{\lambda}_3^i = \left(0, \quad 0, \quad \frac{1}{r\sin\theta}, \quad 0 \right) \\ \bar{\lambda}_4^i = \left(0, \quad 0, \quad 0, \quad \frac{1}{r^{\frac{n}{2}}} \right) \end{cases}$$

the physical components $R_{(abcd)}$ of the curvature tensor are

$$(2.36)\begin{cases} R_{1212} = \frac{(an+a+1)E}{2r^{an+a+3}}, \\ R_{1313} = -\frac{(an+a+1)E}{2r^{an+a+3}}, \\ R_{1414} = \frac{2n-n^2}{4r^2}\left(\frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}} \right) + \frac{nE(an+a+1)}{4r^{a(n+1)+3}}, \\ R_{2323} = -\frac{1}{r^2}\left(\frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}} \right), \\ R_{2424} = -\frac{n}{2r^2}\left(\frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}} \right), \\ R_{3434} = -\left(\frac{n}{2r^2} \right)\left(\frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}} \right). \end{cases}$$

We see that $R_{(abcd)} \to 0$ as $r \to \infty$. It follows that the space-time is asymptotically homaloidal.

Also the metric (2.33) the fluid velocity u^i is given by

$$(2.37)\quad u^1 = u^2 = u^3 = 0 = u_1 = u_2 = u_3 \quad \text{and} \quad u_4 = r^{n/2}, u^4 = \frac{1}{r^{n/2}}.$$

The scalar of expansion $\Theta = u^i_{;i}$ is identically zero. The non vanishing components of the tensor of rotation ω_{ij} are

(2.38) $\omega_{14} = -\omega_{41} = -\frac{n}{2} r^{\frac{n}{2}-1}$

The non zero components of the shear tensor σ_{ij} are

(2.39) $\sigma_{14} = \sigma_{41} = \frac{n}{2} r^{\frac{n}{2}-1}$.

3. SOLUTION FOR THE PERFECT FLUID CORE

Pressure and density for the metric (2.33) are

(3.1) $$8\pi p = \frac{8\pi\rho}{a} = \frac{n+1}{r^2}\left[\frac{a+1}{an+a+1} + \frac{E}{r^{an+a+1}}\right] - \frac{1}{r^2}$$

It follows from (3.1) that the density of the distribution tends to infinity as r tends to zero. In order to get rid of the singularity at r=0 in the density we visualize that the distribution has a core of radius r_o and constant density ρ_o. The field inside the core is given by Schwarzschild internal solution.

(3.2) $$\begin{cases} e^{-\lambda} = 1 - \frac{r^2}{R^2}, e^{\nu} = \left[\bar{A} - \bar{B}\left(1 - \frac{r^2}{R^2}\right)\right]^2 \\ 8\pi p = \frac{1}{R^2} \cdot \frac{3\bar{B}\left(1 - \frac{r^2}{R^2}\right) - \bar{A}}{\bar{A} - \bar{B}\left(1 - \frac{r^2}{R^2}\right)^{1/2}} \end{cases}$$

Where $\bar{A}$ and $\bar{B}$ are constants and $R^2 = \frac{3}{8\pi\rho}$.

The continuity conditions for the metric (2.33) and (3.2) at the boundary gives

$$(3.3)\begin{cases} R^2 = \dfrac{r_o^2}{\left(\dfrac{an}{an+a+1} - \dfrac{E}{r_o^{an+a+1}}\right)}, \\ \bar{A} = r_o^{n/2} + \dfrac{nR^2}{2r_o^{2-\frac{n}{2}}}\left(1 - \dfrac{r_o^2}{R^2}\right) \\ \bar{B} = \dfrac{nR^2}{2r_o^{2-\frac{n}{2}}}\left(1 - \dfrac{r_o^2}{R^2}\right)^{1/2}, \\ E = r_o^{an+a+1}\left(\dfrac{an}{an+a+1} - \dfrac{r_o^2}{R^2}\right). \end{cases}$$

and the density of the core

$$(3.4)\quad \rho_o = \frac{3}{8\pi r^2}\left(\frac{an}{an+a+1} - \frac{E}{r^{an+a+1}}\right).$$

which complete the solution for the perfect fluid core of radius r_o surrounded by considered fluid. The energy condition $T_{ij}\, U^i\, U_j > 0$ and the Hawking and Penrose condition (Hawking and Penrose, 1970).

$$\left(T_{ij} - \frac{1}{2} g_{ij} T\right) u^i u_j > 0,$$

both reduces to $\rho > 0$ which is obviously satisfied.

For different values of a and n, solutions obtained above in case II provide many previously known solutions. For a=1, n=1, we get the results due to Yadav and Saini [20]. For n=2 and by suitable adjustment of constant we get the solution due to Singh and Yadav [23]. Also for a=3 and n=2 we get the solution due to Yadav and Purushottom [21] and Yadav et al [22] by suitable adjustment of constants.

4. DISCUSSION

In this paper the equation of state for the fluid has been taken as p=aρ which (for a=1) describes several important cases, e.g. relativistic degenerate Fermi gas and probably very dense baryon matter (Zeldovich and Novikov [26]; Walecka [8]). The casual limit for ideal gas has also form ρ=p (Zeldovich and Novkove [26]).

Furthermore, if the fluid satisfies the equation of state p=ρ and if in addition its motion is irrotational, then such a source has the same stress energy tensor as that of a m assless scalar field (Tabensky and Taub [19]). Also the solution in this case can be transformed to the solution of Brans-Dicke Theory in vacuum. (Dicke [17]).

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