International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 2 Issue 5, May - 2013

Static Analysis of Laminated Composite Circular and Annular Plates

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Abstract

Analytical solutions are presented for the static deflection analysis of laminated polar orthotropic circular and annular plates. The analysis is based on the application of the first order shear deformation theory employed. Three linear partial differential equations for axisymmetric deformations are written in terms of displacements u, α and w. Chebyshev collocation method is employed for the solution of the evaluation of static deflection problem. Numerical results are presented to show the validity and accuracy of the proposed method. Results of parametric studies conducted to evaluate the effect of parameters like orthotropic ratios, number of layers, lamination sequences and boundary conditions, on the response of laminated polar orthotropic circular and annular plates are also presented.

1. Introduction

Fiber reinforced laminated composites are being increasingly used in modern engineering applications due to their high specific strength and high specific modulus. The increased application of laminated composites in the primary components in structures like spacecrafts, high speed aircrafts, missiles, gas turbines, etc. are due to the number of advantages they offer in structural, operational, production and/or maintenance aspects.

The use of advanced composite materials for structural elements brings in the need to develop new analytical and design techniques. With the present level of their application being what it is, it becomes a necessity to develop better mathematical models to predict the mechanical behavior of structural elements made up of such materials, under service loads.

In the present work, it is proposed to study, the static deflection analysis of laminated polar orthotropic circular and annular plates by Chebyshev collocation method. A first order shear deformation theory is used in terms of u, α and w. These field variables are expanded in polynomials and then orthogonal point collocation method is used to discretise the governing equations. To demonstrate the convergence of the method, numerical results are presented for clamped and simply supported isotropic and polar orthotropic circular and annular plates. The validity of the analytical solution is confirmed by comparing with data obtained from open literature.

2. Mathematical formulation

The laminated plate of constant thickness h is composed of polar orthotropic laminae stacking symmetrically or anti-symmetrically about the middle surface of plate. Plate co-ordinates (r, θ, z) used are as shown in figure 1, where u, v, w denote the displacements of any point of the plate in the corresponding r, θ, z directions.

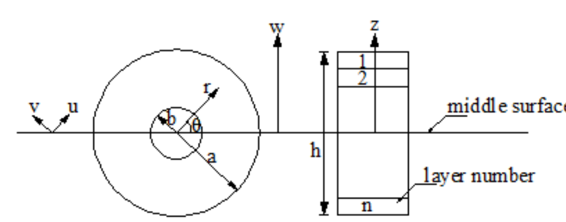

Figure 2.1 Geometry of a n-layered laminate

In this study, first order shear deformation theory is employed and the general displacement field is assumed in the form

$$u(r, \theta, z) = u^{o}(r, \theta) + z\,\alpha_1(r, \theta),$$
$$v(r, \theta, z) = v^{o}(r, \theta) + z\,\alpha_2(r, \theta),$$
$$w(r, \theta, z) = w(r, \theta) \qquad (1)$$

where u^o, v^o, w denote the displacements of any point on the middle surface and α_1, α_2 are the rotations of the normal to the mid-plane about θ, r axes respectively.
Strain displacement relations are of the following form in polar co-ordinates

$$\varepsilon_r = \overset{\circ}{\varepsilon}_r + z\cdot\kappa_r,\quad \varepsilon_\theta = \overset{\circ}{\varepsilon}_\theta + z\cdot\kappa_\theta,$$
$$\gamma_{r\theta} = \overset{\circ}{\gamma}_{r\theta} + z\cdot\kappa_{r\theta},\quad \gamma_{rz} = \overset{\circ}{\gamma}_{rz},\quad \gamma_{\theta z} = \overset{\circ}{\gamma}_{\theta z} \qquad (2)$$

where the reference surface strains and curvatures are given by,

$$\overset{\circ}{\varepsilon}_r = \frac{\partial u^\circ}{\partial r},\qquad \overset{\circ}{\varepsilon}_\theta = \frac{1}{r}\cdot\left(\frac{\partial v^\circ}{\partial\theta} + u^\circ\right),$$
$$\overset{\circ}{\gamma}_{r\theta} = \frac{1}{r}\cdot\left(\frac{\partial u^\circ}{\partial\theta} - v^\circ\right) + \frac{\partial v^\circ}{\partial r},$$
$$\overset{\circ}{\gamma}_{rz} = \alpha_1 + \frac{\partial w}{\partial r},\qquad \overset{\circ}{\gamma}_{\theta z} = \alpha_2 + \frac{1}{r}\cdot\left(\frac{\partial w}{\partial\theta}\right),$$
$$\kappa_r = \frac{\partial\alpha_1}{\partial r},\qquad \kappa_\theta = \frac{1}{r}\cdot\left(\frac{\partial\alpha_2}{\partial\theta} + \alpha_1\right),$$
$$\kappa_{r\theta} = \frac{1}{r}\cdot\left(\frac{\partial\alpha_1}{\partial\theta} - \alpha_2\right) + \frac{\partial\alpha_2}{\partial r} \qquad (3)$$

According to the shear deformation theory, the constitutive equations for the k^{th} layer of a polar orthotropic laminated plate can be written in the following form in polar co-ordinates.

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{Bmatrix} = \begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\\ \gamma_{r\theta}\end{Bmatrix},$$
$$\begin{Bmatrix}\tau_{rz}\\ \tau_{\theta z}\end{Bmatrix} = \begin{bmatrix}C_{44} & 0\\ 0 & C_{55}\end{bmatrix}\begin{Bmatrix}\gamma_{rz}\\ \gamma_{\theta z}\end{Bmatrix} \qquad (4)$$

where $C_{11} = \frac{E_r}{1-\upsilon_r\upsilon_\theta}$, $C_{12} = \frac{\upsilon_r E_\theta}{1-\upsilon_r\upsilon_\theta} = \frac{\upsilon_\theta E_r}{1-\upsilon_r\upsilon_\theta}$,

$$C_{22} = \frac{E_\theta}{1-\upsilon_r\upsilon_\theta}$$

$$C_{44} = G_{rz},\quad C_{55} = G_{\theta z},\quad C_{66} = G_{r\theta} \qquad (5)$$

Where E_r and E_θ are Young's moduli of elasticity in r and θ directions. υ_r and υ_θ are Poisson's ratios in r and θ directions. $G_{r\theta}$, $G_{\theta z}$ and G_{rz} are the shear moduli in the respective planes.
The stress resultants acting on a laminate are obtained as:

$$\begin{bmatrix}N_r & M_r\\ N_\theta & M_\theta\\ N_{r\theta} & M_{r\theta}\end{bmatrix} = \int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{Bmatrix}(1, z)\, dz,$$
$$\begin{Bmatrix}Q_r\\ Q_\theta\end{Bmatrix} = \int_{-h/2}^{h/2}\begin{Bmatrix}\tau_{rz}\\ \tau_{\theta z}\end{Bmatrix}dz \qquad (6)$$

where z is the distance of the lamina from the middle plane.
Substituting the stress strain relations, we have the constitutive matrix as

$$\begin{Bmatrix}N_r\\ N_\theta\\ N_{r\theta}\\ M_r\\ M_\theta\\ M_{r\theta}\\ Q_r\\ Q_\theta\end{Bmatrix} = \begin{bmatrix}A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 & 0 & 0\\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 & 0 & 0\\ 0 & 0 & A_{66} & 0 & 0 & B_{66} & 0 & 0\\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 & 0 & 0\\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 & 0 & 0\\ 0 & 0 & B_{66} & 0 & 0 & D_{66} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & K^2A_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K^2A_{55}\end{bmatrix}\begin{Bmatrix}\overset{\circ}{\varepsilon}_r\\ \overset{\circ}{\varepsilon}_\theta\\ \overset{\circ}{\gamma}_{r\theta}\\ \kappa_r\\ \kappa_\theta\\ \kappa_{r\theta}\\ \overset{\circ}{\gamma}_{rz}\\ \overset{\circ}{\gamma}_{\theta z}\end{Bmatrix} \qquad (7)$$

where, $A_{ij} = \sum_{k=1}^{n}(C_{ij})_k(z_k - z_{k-1})$,
i, j = 1, 2,6,4,5

$B_{ij} = \frac{1}{2}\sum_{k=1}^{n}(C_{ij})_k(z_k^2 - z_{k-1}^2)$,
i, j = 1, 2,6

$D_{ij} = \frac{1}{3}\sum_{k=1}^{n}(C_{ij})_k(z_k^3 - z_{k-1}^3)$,
where i, j = 1, 2,6 (8)

A_{ij} are the extensional stiffnesses, B_{ij} are the bending-extension coupling stiffnesses, D_{ij} are the bending stiffnesses and K^2 is the Shear correction factor introduced to account for non-uniform distribution of the transverse shear strains through the thickness of the plate, which is taken as $\pi^2/12$.
n is the total number of layers in the laminate.
The stress resultants and stress couples defined in (6) and (7) must satisfy the following equilibrium equations **(Ravichandran, 1989)**

$$\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial\theta} + \frac{(N_r - N_\theta)}{r} = 0,$$
$$\frac{\partial N_{r\theta}}{\partial r} + \frac{\partial N_\theta}{r\,\partial\theta} + 2\frac{N_{r\theta}}{r} = 0$$
$$\frac{\partial Q_r}{\partial r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial\theta} + \frac{Q_r}{r} + q(r,\theta) = 0,$$
$$\frac{\partial M_r}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta} + \frac{(M_r - M_\theta)}{r} = Q_r$$

$$\frac{\partial M_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial M_{\theta}}{\partial \theta}+2\frac{M_{r\theta}}{r}=Q_{\theta} \tag{9}$$

If the analysis is now restricted to only axisymmetric case, the deformations are symmetrical about z-axis, thus the stresses and strains are independent of θ and

$\tau_{r\theta}=\tau_{z\theta}=0$, $v^o=0$, $\alpha_2=0$ and also $\frac{\partial}{\partial\theta}(\ )=0$

The governing equations for axi-symmetric case then can be written in terms of mid-plane displacements as,

$$A_{11}\left(\frac{\partial^2 u^\circ}{\partial r^2}+\frac{\partial u^\circ}{r\cdot\partial r}\right)-A_{22}\frac{u^\circ}{r^2}+B_{11}\left(\frac{\partial^2\alpha_1}{\partial r^2}+\frac{\partial\alpha_1}{r\cdot\partial r}\right)-B_{22}\frac{\alpha_1}{r^2}=0$$

$$K^2A_{44}\left(\frac{\partial\alpha_1}{\partial r}+\frac{\alpha_1}{r}+\frac{\partial^2 w^o}{\partial r^2}+\frac{\partial w^o}{r\cdot\partial r}\right)+q(r)=0$$

$$B_{11}\left(\frac{\partial^2 u^\circ}{\partial r^2}+\frac{\partial u^\circ}{r\cdot\partial r}\right)-B_{22}\frac{u^\circ}{r^2}+D_{11}\left(\frac{\partial^2\alpha_1}{\partial r^2}+\frac{\partial\alpha_1}{r\cdot\partial r}\right)$$

$$-D_{22}\frac{\alpha_1}{r^2}=K^2A_{44}\left(\alpha_1+\frac{\partial w^o}{\partial r}\right) \tag{10}$$

For convenience, the following dimensionless parameters are introduced

$$U=\frac{u^\circ\cdot(a-b)}{h^2},\quad W=\frac{w^o}{h},\quad \alpha=\frac{\alpha_1\cdot(a-b)}{h},$$

$$\xi=\frac{r-b}{a-b},\quad p=\frac{q\cdot(a-b)^4}{E_T\cdot h^4},\quad a_{44}=\frac{K^2\cdot A_{44}}{E_T\cdot h},$$

$$a_{ij}=\frac{A_{ij}}{E_T\cdot h},\quad b_{ij}=\frac{B_{ij}}{E_T\cdot h^2},\quad d_{ij}=\frac{D_{ij}}{E_T\cdot h^3} \tag{11}$$

where i, j = 1, 2,6

Here, E_T is the reference Young's modulus. In case of laminated composites with layers of same material, E_T is taken to be the Young's modulus in the direction transverse to fiber direction.

Thus the governing differential equations of the plate can now be expressed in dimensionless form as,

$$a_{11}\left(\frac{1}{(a-b)^2}\cdot\frac{\partial^2 U}{\partial\xi^2}+\frac{1}{(\xi a+(1-\xi)b)*(a-b)}\cdot\frac{\partial U}{\partial\xi}\right)-a_{22}\frac{U}{(\xi a+(1-\xi)b)^2}+$$

$$b_{11}\left(\frac{1}{(a-b)^2}\cdot\frac{\partial^2\alpha}{\partial\xi^2}+\frac{1}{(\xi a+(1-\xi)b)*(a-b)}\cdot\frac{\partial\alpha}{\partial\xi}\right)-b_{22}\frac{\alpha}{(\xi a+(1-\xi)b)^2}=0$$

$$a_{44}\left(\frac{1}{(a-b)}\cdot\frac{\partial\alpha}{\partial\xi}+\frac{\alpha}{(\xi a+(1-\xi)b)}+\frac{1}{(a-b)}\cdot\frac{\partial^2 W}{\partial\xi^2}+\frac{1}{(\xi a+(1-\xi)b)}\cdot\frac{\partial W}{\partial\xi}\right)$$

$$+p\cdot\frac{h^2}{(a-b)^3}=0$$

$$b_{11}\left(\frac{1}{(a-b)^2}\cdot\frac{\partial^2 U}{\partial\xi^2}+\frac{1}{(\xi a+(1-\xi)b)*(a-b)}\cdot\frac{\partial U}{\partial\xi}\right)-b_{22}\frac{U}{(\xi a+(1-\xi)b)^2}+$$

$$d_{11}\left(\frac{1}{(a-b)^2}\cdot\frac{\partial^2\alpha}{\partial\xi^2}+\frac{1}{(\xi a+(1-\xi)b)*(a-b)}\cdot\frac{\partial\alpha}{\partial\xi}\right)-d_{22}\frac{\alpha}{(\xi a+(1-\xi)b)^2}$$

$$=\frac{a_{44}}{h^2}\left(\alpha+\frac{\partial W}{\partial\xi}\right) \tag{12}$$

2.1 Polynomial series solution by collocation at Chebyshev zeroes

The Chebyshev polynomials $T_n(x)$ are a class of orthogonal polynomials, which are defined as,

$T_n(x)=\cos n\theta$

Where $\cos\theta=x \quad , \quad -1\le x\le 1$ (13)

These provide a solution of the second order differential equation,

$$(1-x^2)\,T_n''-x\cdot T_n'+n^2\,T_n=0 \tag{14}$$

where $(\ )'=\frac{d}{dx}(\ )$

In many applications, it is advantageous to re-define the polynomials in the range $0\le\xi\le 1$. The shifted Chebyshev polynomial $T_r^*(\xi)$ of degree r in the range $0\le\xi\le 1$ is derived from the polynomial $T_r(x)$ $\{-1\le x\le 1\}$ using a linear transformation,

$$\xi=\frac{1}{2}\{x+1\} \tag{15}$$

i.e., $T_r(\xi)=\cos rt$ where $\cos t=(2\xi-1)$ (16)

The shifted polynomials satisfy the recurrence relations

$$T_{r+1}^*(\xi)\ +\ T_{r-1}^*(\xi)=2\,(2\xi-1)\,T_r^*(\xi) \tag{17}$$

and the orthogonality conditions

$$\int_0^1\frac{T_m^*(\xi)\ T_n^*(\xi)}{\sqrt{\xi}\ \sqrt{(1-\xi)}}\,d\xi\ =\ \begin{cases}0 & \text{for}\quad m\ne n\\ \pi/2 & \text{for}\quad m=n\ne 0\\ \pi & \text{for}\quad m=n=0\end{cases} \tag{18}$$

i.e., the shifted Chebyshev polynomials are orthogonal with respect to the weighing function

$$W(\xi)=\frac{1}{\{\xi(1-\xi)\}^{1/2}} \tag{19}$$

Any continuous function $g(\xi)$ in the interval $0\le\xi\le 1$, can be represented by a infinite series of the form

$$g(\xi)\ =\ \frac{g_o}{2}T_o^*(\xi)+\sum_{r=1}^{\infty}g_rT_r^*(\xi) \tag{20}$$

where g_i (i=0,1,2,….) are the coefficients to be determined so as to obtain a best possible fit. The series in eqn.(20) is a fast converging one and good

approximation is obtained by taking a finite number of terms in the above series.
i.e.,

$$g(\xi) = \frac{g_o}{2} T_o^*(\xi) + \sum_{r=1}^{N} g_r T_r^*(\xi) = \sum_{r=0}^{N}{}^{+} g_r T_r^*(\xi) \quad (21)$$

Where, the plus sign indicates that the first term of the series is to be halved.
For a known function $g(\xi)$, using the orthogonality conditions (eqn.18), the coefficients g_r can be calculated as

$$g_r = \frac{2}{\pi} \int_0^1 \frac{g(\xi)\, T_r^*(\xi)}{\sqrt{\xi}\,\sqrt{(1-\xi)}}\, d\xi \quad , \; 0 \le r \le N \quad (22)$$

The shifted Chebyshev polynomial $T_r^*(\xi)$ of degree r has r zeroes in the range $0 \le \xi \le 1$, which are used as the collocation points in the present study.
The (N-1) collocation points are taken at the zeroes of the Chebyshev polynomial. For static loads, the radial displacement, deflection and rotation are expanded as polynomials in ξ.

$$\{U(\xi),\ \alpha(\xi),\ W(\xi)\} = \sum_{n=1}^{N+1} \{U_n, \alpha_n, W_n\} \xi^{n-1} \quad (23)$$

Substituting eqn. (23) in eqn. (12), the following equations are obtained.

$$a_{11} \sum_{n=1}^{N+1} \left(\frac{(n-1)(n-2)U_n \xi^{n-3}}{(a-b)^2} + \frac{(n-1)U_n \xi^{n-2}}{(\xi a + (1-\xi)b) * (a-b)} \right) - a_{22} \sum_{n=1}^{N+1} \frac{U_n \xi^{n-1}}{(\xi a + (1-\xi)b)^2} +$$
$$b_{11} \sum_{n=1}^{N+1} \left(\frac{(n-1)(n-2)\alpha_n \xi^{n-3}}{(a-b)^2} + \frac{(n-1)\alpha_n \xi^{n-2}}{(\xi a + (1-\xi)b) * (a-b)} \right) - b_{22} \sum_{n=1}^{N+1} \frac{\alpha_n \xi^{n-1}}{(\xi a + (1-\xi)b)^2} = 0 \quad (24)$$

$$a_{44} \sum_{n=1}^{N+1} \left(\frac{(n-1)\alpha_n \xi^{n-2}}{(a-b)} + \frac{\alpha_n \xi^{n-1}}{(\xi a + (1-\xi)b)} + \frac{(n-1)(n-2)W_n \xi^{n-3}}{(a-b)} + \frac{(n-1)W_n \xi^{n-2}}{(\xi a + (1-\xi)b)} \right) + p \cdot \frac{h^2}{(a-b)^3} = 0 \quad (25)$$

$$b_{11} \sum_{n=1}^{N+1} \left(\frac{(n-1)(n-2)U_n \xi^{n-3}}{(a-b)^2} + \frac{(n-1)U_n \xi^{n-2}}{(\xi a + (1-\xi)b) * (a-b)} \right) - b_{22} \sum_{n=1}^{N+1} \frac{U_n \xi^{n-1}}{(\xi a + (1-\xi)b)^2}$$
$$+ d_{11} \sum_{n=1}^{N+1} \left(\frac{(n-1)(n-2)\alpha_n \xi^{n-3}}{(a-b)^2} + \frac{(n-1)\alpha_n \xi^{n-2}}{(\xi a + (1-\xi)b) * (a-b)} \right) - d_{22} \sum_{n=1}^{N+1} \frac{\alpha_n \xi^{n-1}}{(\xi a + (1-\xi)b)^2}$$
$$- \frac{a_{44}}{h^2} \sum_{n=1}^{N+1} \left(\alpha_n \xi^{n-1} + (n-1) W_n \xi^{n-2} \right) = 0 \quad (26)$$

There are 3(N+1) constants to be determined. The stipulation of the three boundary conditions at each edge provide six equations. 3(N-1) additional equations are obtained by forcing the satisfaction of each of the three differential equations at the (N-1) zeroes of $T^*_{(N-1)}(\xi)$, $0 \le \xi \le 1$ - the $(N\text{-}1)^{th}$ degree shifted Chebyshev Polynomial.
The N^{th} degree Chebyshev Polynomial T_N^* has N zeroes at

$$\xi_i = \frac{1}{2}\left\{1 + \cos\left[\frac{(2*i-1)\pi}{2N}\right]\right\} \qquad i = 1, 2, \ldots N \quad (27)$$

Thus forcing the satisfaction of each of the three differential equations at the (N-1) zeroes of $T^*_{(N-1)}(\xi)$, $0 \le \xi \le 1$ along with the stipulation of the three boundary conditions at each edge results in a set of simultaneous equations of the form,

$$[L_{11}]\{U\} + [L_{12}]\{\alpha\} + [L_{13}]\{W\} = \{0\}$$
$$[L_{21}]\{U\} + [L_{22}]\{\alpha\} + [L_{23}]\{W\} = \{p\}$$
$$[L_{31}]\{U\} + [L_{32}]\{\alpha\} + [L_{33}]\{W\} = \{0\} \quad (28)$$

Where $\{U\}^T = \{U_1, U_2, \ldots\ldots U_{n+1}\}$

$$\{\alpha\}^T = \{\alpha_1, \alpha_2, \ldots\ldots \alpha_{n+1}\}$$
$$\{W\}^T = \{W_1, W_2, \ldots\ldots W_{n+1}\} \quad (29)$$

It can be seen that [L13] = [0] and [L21] = [0].

These algebraic equations are then solved using LU Decomposition method. **(William H. Press *et al*, 2002)** The coefficients are then substituted in eqn. (23) to get radial displacement, transverse deflection and rotation at any point inside the plate.

3. Convergence and comparison studies

In order to validate the procedure implemented, static analysis has been conducted on isotropic plates. The convergence analysis of the Chebyshev collocation method for static deflection of circular and annular plates is first carried out. **Table 3.1** shows the results of analysis of an isotropic circular plate subjected to axisymmetric pressure q, with increasing number of terms used in each of the series in eqn. (23). It can be observed that sufficiently accurate, converged solution is obtained taking 10 – 12 terms used in each of the series.

Isotropic circular and annular plates subjected to various types of pressure loads are now analyzed. **Table 3.2** and **Table 3.3** shows the results of static deflection and rotation of the transverse normal in radial direction, obtained using the present solution. It can be observed that the present solution provides results comparing reasonably well with the results obtained by CPT [**Warren C. Young (1989)]**. The deviations between the results can be attributed in fact to effect of shear deformations as the plates considered herein are moderately thick plates. (a/h = 10)

Table 3.1 Static deflection of isotropic circular plates - convergence study

a = 100 mm, h=10 mm, υ = 0.3, q = 100 N/mm^2, E_θ = 2.10 kN/mm^2

Boundary condition: clamped edges

$$W = 10^2 \frac{w^o}{h}$$

Number of terms in Chebyshev series N	Maximum deflection W
4	0.8438
6	0.8437
8	0.8438
10	0.8437
12	0.8437

Table 3.2 Static solution for isotropic circular plates

a/h = 10 , υ = 0.3

Type of load	Boundary Condition		Present (FSDT)	Warren C. Young (1989) (CPT)
Uniformly Distributed Load	Clamped	Ky_{max}	0.01622	0.01563
	Simply Supported	Ky_{max}	0.06430	0.06370
		$K_{\theta max}$	0.09615	0.09615
Uniformly Varying Load	Clamped	Ky_{max}	0.00693	0.00667
	Simply Supported	Ky_{max}	0.03257	0.03231
		$K_{\theta max}$	0.05128	0.05128
Parabolically Varying Load	Clamped	Ky_{max}	0.00362	0.00347
	Simply Supported	Ky_{max}	0.01964	0.01949
		$K_{\theta max}$	0.03205	0.03205

$W_{max} = Ky_{max} \frac{q a^4}{D}$ $\alpha_{max} = K_{\theta max} \frac{q a^3}{D}$

Table 3.3 Static solution for isotropic annular plates

a/h = 10 , b/a = 0.1, υ = 0.3

Type of load	Boundary Condition		Present (FSDT)	Reference (CPT)*
Uniformly Distributed Load	C – C	Ky_{max}	0.0020	0.0018
	S^a – C	Ky_{max}	0.0025	0.0025
		$K_{\theta b}$	0.0100	0.0135
	C^a – S	Ky_{max}	0.0044	0.0040
		$K_{\theta a}$	0.0140	0.0147
	S – S	Ky_{max}	0.0061	0.0060
		$K_{\theta b}$	0.0196	0.0264
		$K_{\theta a}$	0.0176	0.0198
Uniformly Varying Load	C – C	Ky_{max}	0.0010	0.0009
	S^a – C	Ky_{max}	0.0013	0.0013
		$K_{\theta b}$	0.0043	0.0059
	C^a – S	Ky_{max}	0.0026	0.0024
		$K_{\theta a}$	0.0087	0.0093

	S – S	Ky_{max}	0.0035	0.0034
		$K_{\theta b}$	0.0114	0.0137
		$K_{\theta a}$	0.0108	0.0119
Parabolically Varying Load	C – C	Ky_{max}	0.0006	0.0005
	S^a – C	Ky_{max}	0.0007	0.0007
		$K_{\theta b}$	0.0025	0.0031
	C^a – S	Ky_{max}	0.0017	0.0016
		$K_{\theta a}$	0.0060	0.0064
	S – S	Ky_{max}	0.0022	0.0022
		$K_{\theta b}$	0.0074	0.0083
		$K_{\theta a}$	0.0073	0.0080

$W_{max} = Ky_{max} \dfrac{q a^4}{D}$, $\alpha = K_\theta \dfrac{q a^3}{D}$ * **Warren C. Young (1989)**

[a] Boundary condition at inner edge of the plate

4. Parametric study

In order to bring out the effect of shear deformation in case of plates undergoing axisymmetric deformations, different shear moduli are considered.

(1) $G_{rz} = \infty$ (4) $G_{rz} = \dfrac{0.5\,E}{1+\upsilon}$

(2) $G_{rz} = \dfrac{0.75\,E}{1+\upsilon}$ (5) $G_{rz} = \dfrac{0.375\,E}{1+\upsilon}$

(3) $G_{rz} = \dfrac{0.625\,E}{1+\upsilon}$ (6) $G_{rz} = \dfrac{0.25\,E}{1+\upsilon}$

Table 4.1 shows the results of analysis of an isotropic plate undergoing axisymmetric deformations. For very large value of shear modulus, the results of the present work are comparable with the results given by **Warren C. Young (1989)**. It can also be observed from this table that the results with different values of shear moduli considered (in the narrow range) are very close to each other and to the value due to CPT, which neglects shear deformation.

Table 4.2 and **Table 4.3** show the results of a study conducted to know the effect of orthotropy ratio on the transverse deflection and radial displacement of laminated polar orthotropic circular and annular plates respectively. The need to select proper shear correction factor K^2 in the analysis makes the first order shear deformation theory of the present work somewhat empirical. There are different recommendations for the selection of shear correction factors. Quite often the value of $\pi^2/12$ recommended for isotropic plates is assumed to be appropriate even in the case of composite plates and the same is followed here. It can be seen that, the increase in orthotropy ratio decreases the displacements for all the lamination schemes considered herein. Also, in case of annular plates, among the different lamination sequences, the least deflection is observed when the laminate is made up of laminae in which fibers are arranged in 0º only. Surprisingly, in case of circular plates with simply supported boundary condition, the least deflection is observed when the laminate is made up of laminae in which fibers are arranged in 90º only. In case of unsymmetric and anti-symmetric laminates, since the coupling rigidity $[B_{ij}]$ exists, radial displacements are also observed.

Table 4.1 Effect of shear deformation on the deflection of plate

a/h = 10, Boundary condition: clamped edges

$\dfrac{G(1+\upsilon)}{E}$	Ky_{max}	
	Present (FSDT)	**(CPT)***
∞	0.01562	0.01563
0.75	0.01602	
0.625	0.01610	
0.5	0.01622	
0.375	0.01642	
0.25	0.01682	

$W_{max} = Ky_{max} \dfrac{q a^4}{D}$ * **Warren C. Young (1989)**

Table 4.2 Effect of orthotropy ratio on displacements of laminated polar orthotropic circular plates

a = 100 mm, h = 10 mm, $\upsilon_{r\theta}$ = 0.31, G_{rz}/E_θ = 0.253, q = 100 N/mm^2 ,

BC	Displacement	E_r/E_θ / Lamination scheme	5	10	15	30
Clamped	W	90º	0.0582	0.0408	0.0327	0.0223
		90º/0º/90º	0.0559	0.0378	0.0297	0.0195
		90º/0º/0º/90º	0.0516	0.0333	0.0256	0.0166
		0º/90º	0.0469	0.0304	0.0234	0.0153
		90º/0º/90º/0º	0.0438	0.0271	0.0206	0.0136
		0º	0.0428	0.0250	0.0191	0.0125
		0º/90º/0º	0.0417	0.0248	0.0190	0.0125
		0º/90º/90º/0º	0.0403	0.0232	0.0186	0.0123
	U	0º/90º	0.0172	0.0056	0.0040	0.0027
		90º/0º/90º/0º	0.0026	0.0018	0.0014	0.0008
Simply Supported	W	0º	0.6379	0.4349	0.4204	0.4090
		0º/90º/90º/0º	0.2270	0.0494	1.2854	0.3627
		0º/90º/0º	0.1846	0.4104	0.3639	0.2728
		0º/90º	0.1933	0.1820	0.1328	0.0796
		90º/0º/90º/0º	0.1812	0.1063	0.0759	0.0425
		90º/0º/0º/90º	0.1459	0.0835	0.0597	0.0340
		90º/0º/90º	0.1418	0.0815	0.0585	0.0336
		90º	0.1404	0.0810	0.0584	0.0337
	U	0º/90º	0.0914	0.0498	0.0334	0.0193
		90º/0º/90º/0º	0.0098	0.0074	0.0057	0.0033

$$W = \frac{w^o}{h} \qquad U = \frac{u^o \cdot a}{h^2}$$

Table 4.3 Effect of orthotropy ratio on displacements of laminated polar orthotropic annular plates

a = 100 mm, h = 10 mm, b = 10 mm, $\upsilon_{r\theta}$ = 0.31, G_{rz}/E_θ = 0.253, q = 100 N/mm^2

BC	Displacement	E_r/E_θ / Lamination scheme	5	10	15	30
C – C	W	90º	1.2031	1.0107	0.8956	0.7150
		90º/0º/90º	1.1303	0.9063	0.7823	0.6038
		90º/0º/0º/90º	0.9981	0.7600	0.6437	0.4938
		0º/90º	0.8386	0.6513	0.5634	0.4489
		90º/0º/90º/0º	0.7408	0.5472	0.4704	0.3831
		0º/90º/90º/0º	0.5934	0.4470	0.3941	0.3375
		0º/90º/0º	0.5746	0.4354	0.3857	0.3327
		0º	0.5673	0.4309	0.3825	0.3309
	U	0º/90º	0.2743	0.2283	0.1888	0.1236
		90º/0º/90º/0º	0.1108	0.0796	0.0610	0.0360
S - S	W	90º	3.9604	1.7014	1.4370	1.0204
		90º/0º/90º	2.9180	1.6621	1.2933	0.9317
		90º/0º/0º/90º	2.3716	1.5949	1.2860	0.8963
		0º/90º	1.7369	1.1620	0.9406	0.6936
		90º/0º/90º/0º	1.7061	1.1210	0.9014	0.6576
		0º/90º/90º/0º	1.3923	0.9238	0.7560	0.5806
		0º/90º/0º	1.3455	0.8932	0.7346	0.5722
		0º	1.3255	0.8814	0.7265	0.5692
	U	0º/90º	0.3120	0.3015	0.2637	0.2036
		90º/0º/90º/0º	0.1333	0.1162	0.0954	0.0664

$$W = 10^2 \frac{w^o}{h} \qquad U = 10^2 \frac{u^o \cdot a}{h^2}$$

5. Conclusions

Studies have been conducted to understand the static deflection characteristics of composite circular and annular plates by the Chebyshev collocation technique using first order shear deformation theory. Convergence tests were conducted for the Chebyshev collocation technique and it can be seen that there is excellent convergence even when we take four or six terms in the series for the problems considered. Further, numerical results have been worked out for isotropic and polar orthotropic laminated circular and annular plates with different combinations of clamped and simply supported boundary conditions.

From the results it can be seen that, the increase in orthotropy ratio decreases the displacements for all the lamination schemes considered. Also, in case of annular plates, among the different lamination sequences, the least deflection is observed when the laminate is made up of laminae in which fibers are arranged in 0º only. Surprisingly, in case of circular plates with simply supported boundary condition, the least deflection is observed when the laminate is made up of laminae in which fibers are arranged in 90º only. In case of unsymmetric and anti-symmetric laminates, since the coupling rigidity [B_{ij}] exists, radial displacements are also observed.

6. References

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