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Steady State Analysis Of A Non-Markovian Bulk Queueing Model With Multiple Vacation , Accessible Batches, Setup Times With N-Policy And Closedown Times

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Abstract

In this paper, a generalized non-Markovian bulk arrival service queueing system is considered with multiple vacations, setup times and closedown time. The service starts only if minimum of 'a' customers are available in the queue. At the service completion epoch, if the number of customers is ξ ,where a≤ ξ ≤d-1 (d≤b) then the server takes the entire queue for batch service and admits the subsequent arrivals for service till the service of the current batch is over, or the accessible limit d is reached, whichever occurs first. At the service initiation epoch, if the number of customers waiting in the queue 'ξ' is atleast 'd' (a≤d≤b), then the server takes min(ξ ,b) customers for service and does not allow further arrival into the batch. On completion of a service, if the queue length is less than 'a', then the server performs a closedown work such as, shutting down the machine, removing the tools etc. Following closedown work, the server leaves for a vacation of random length irrespective of queue length. When the server returns for a vacation and if the queue length is still less than 'a', he leaves for another vacation and so on until he finds atleast 'a' customers waiting for service in the queue. That is, if the server finds atleast 'N' customers waiting for service, then he requires a setup time 'R' to start the service. After the setup he serves a batch of 'b' customers, where b≥a. Various characteristics of queueing system and a cost model are presented.

Keywords**:** Markovian Bulk, Multiple Vacation, Setup time, Accessible Batches and Closedown times.

1. Introduction

Many researchers have concentrated on bulk service queueing models, in which once the service is started arriving customers, can not enter the service station though enough space is available to accommodate them. It can be observed in many practical situations that arriving customers will be considered for service with current batch in service with some restriction. The concept of non accessibility while receiving service, has been studied by Weiss[6], Sivasamy[7] analyzed a Markovian single arrival bulk service queue with accessible and non accessible batches. R. Arumuganathan and S. Jeyakumar[1] had given results for setup times with N-Policy. Sharma and Jain[8] obtained results for average queue length and waiting time distribution for state dependent Markovian single arrivl bulk service queueing system with accessible and non accessible batches. Sharma et al.[9] established the expression for average queue length for state dependent Mx/M(a,d,b)/1 queue with accessible and non accessible batches without vacations. In the literature, only less attention is given for general because of the complexity in getting a closed form solution.

In this paper, a generalized non-Markovian bulk arrival service queueing system is considered with multiple vacations, setup times and closedown time. The service starts only if minimum of 'a' customers are available in the queue. At the service completion epoch, if the number of customers is ξ ,where a≤ ξ ≤d-1 (d≤b) then the server takes the entire queue for batch service and admits the subsequent arrivals for service till the service of the current batch is over, or the accessible limit d is reached, whichever occurs first. At the service initiation epoch, if the number of customers waiting in the queue 'ξ' is atleast 'd' ($a \leq d \leq b$), then the server takes min(ξ,b) customers for service and does not allow further arrival into the batch. On completion of a service, if the queue length is less than 'a', then the server performs a closedown work such as, shutting down the machine, removing the tools etc. Following closedown work, the server leaves for a vacation of random length irrespective of queue length. When the server returns for a vacation and if the queue length is still less than 'a', he leaves for another vacation and so on until he finds atleast 'a' customers waiting for service in the queue. That is, if the server finds atleast 'N' customers waiting for service, then he requires a setup time 'R' to start the service. After the setup he serves a batch of 'b' customers, where $b \geq a$.

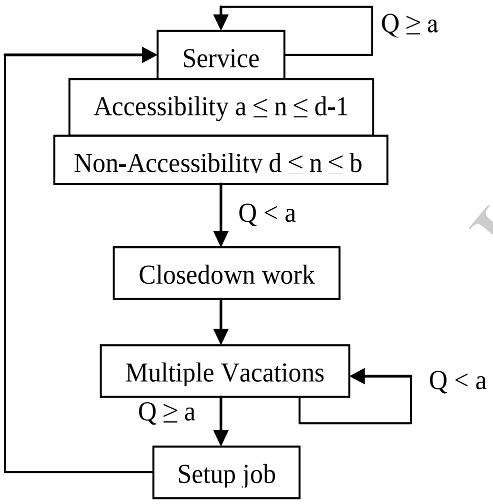

Fig 1: Schematic Representation of Queueing Model

For the proposed model, the probability generating function of the number of customers in the queue at an arbitrary time epoch is obtained using supplementary variable technique. The complexity of general service accessible batch queueing system involving LST of unknown probability functions is overcome by providing a recursive epoch. Expression for expected queue length, expected length of idle period, expected length of busy period and expected waiting time are derived. A cost model of the queueing system is discussed.

2. Mathematical model

Let X be the group size random variable, λ be the Poisson arrival rate, g_k be the probability that 'k' customers arrive in a batch and X(z) be its probability generating function. Let S(.),V(.), R(.) and C(.) be the cumulative distributions of the service time, vacation time, setup time and closedown time, respectively. Let s(x), v(x), r(x) and c(x) be the probability density functions of service time, vacation time, setup time and closedown time respectively. $S^0(t)$, $V^0(t)$, $R^0(t)$ and $C^0(t)$ denote the remaining service tine of a batch, the remaining vacation time, setup time and closedown time of a server at an arbitrary time t, respectively. Let us denote the Laplace transform (LT)of s(x), v(x), r(x) and c(x) as $\tilde{S}$, $\tilde{V}$, $\tilde{R}$ and $\tilde{C}$ respectively.

The number of customers in the queue and the number of customers in service are denoted by $N_s(t)$, $N_q(t)$, respectively. The different states of the server at time 't' are defined as follows:

Y(t) = 0, if the server is busy with bulk service
= 1, if the server is doing closedown work
= 2, if the server is on vacation

and define Z(t) = j, if the server is on j^{th} vacation starting from the idle period.

To obtain system equations, the following probabilities are defined. Let

$P_{ij}(x,t)dt = P\{N_s(t)=i,\ N_q(t)=j,\ x \leq S^0(t) \leq x+dt,\ Y(t)=0\},\quad a \leq i \leq b,\ j \geq 0,$

which means that there are i customers under service, j customers in the queue, the server is busy with remaining service time of x.

In a similar manner, it is defined,

$C_n(x,t)dt = P\{N_q(t)=n,\ x \leq C^0(t) \leq x+dt,\ Y(t)=1\},\quad n \geq 0,$

$Q_{jn}(x,t)dt = P\{N_q(t)=n,\ x \leq V^0(t) \leq x+dt,\ Y(t)=2,\ Z(t)=j\},\quad n \geq 0,\ j \geq 1,$

$R_n(x,t)dt = P\{N_q(t)=n,\ x \leq R^0(t) \leq x+dt,\ Y(t)=1\},\quad n \geq N$

3. Analysis

The steady state queue size equations are obtained as

$$-P_{i0}'(x) = -\lambda P_{i0}(x) + \sum_{m=d}^{b} P_{mi}(0)s(x) + \sum_{l=1}^{\infty} Q_{li}(0)s(x) + \sum_{k=1}^{i-a} P_{i-k,0}(x)\lambda g_k, \quad a \leq i \leq d \qquad \rightarrow (1)$$

$$-P_{i0}^{'}(x) = -\lambda P_{i0}(x) + \sum_{m=d}^{b} P_{mi}(0)s(x) + \sum_{l=1}^{\infty} Q_{li}(0)s(x), \quad d+1 \le i \le b \quad \rightarrow (2)$$

$$-P_{dn}^{'}(x) = -\lambda P_{dn}(x) + \sum_{k=1}^{n} P_{d,n-k}(x)\lambda g_k + \sum_{k=1}^{d-a} P_{d-k,0}(x)\lambda g_{k+n}, \quad n \ge 1 \quad \rightarrow (3)$$

$$-P_{in}^{'}(x) = -\lambda P_{in}(x) + \sum_{k=1}^{n} P_{i,n-k}(x)\lambda g_k, \quad d < i < b-1, n \ge 1 \quad \rightarrow (4)$$

$$-P_{bn}^{'}(x) = -\lambda P_{bn}(x) + \sum_{m=d}^{b} P_{m,b+n}(0)s(x) + \sum_{l=1}^{\infty} Q_{l,b+n}(0)s(x) + \sum_{k=1}^{n} P_{b,n-k}(x)\lambda g_k, \quad n \ge 1 \quad \rightarrow (5)$$

$$-C_0^{'}(x) = -\lambda C_0(x) + \sum_{m=a}^{b} P_{m0}(0)C(x) \quad \rightarrow (6)$$

$$-C_n^{'}(x) = -\lambda C_n(x) + \sum_{m=d}^{b} P_{mn}(0)C(x) + \sum_{k=1}^{n} C_{n-k}(x)\lambda g_k, \quad 1 \le n \le a-1 \quad \rightarrow (7)$$

$$-C_n^{'}(x) = -\lambda C_n(x) + \sum_{k=1}^{n} C_{n-k}(x)\lambda g_k, \quad n \ge a \quad \rightarrow (8)$$

$$-Q_{10}^{'}(x) = -\lambda Q_{10}(x) + C_0(0)v(x) \quad \rightarrow (9)$$

$$-Q_{1n}^{'}(x) = -\lambda Q_{1n}(x) + C_n(0)v(x) + \sum_{k=1}^{n} Q_{1,n-k}(x)\lambda g_k, \quad n \ge 1 \quad \rightarrow (10)$$

$$-Q_{j0}^{'}(x) = -\lambda Q_{j0}(x) + Q_{j-1,0}(0)v(x), \quad j \ge 2 \quad \rightarrow (11)$$

$$-Q_{jn}^{'}(x) = -\lambda Q_{jn}(x) + Q_{j-1,n}(0)v(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda g_k, \quad 1 \le n \le a-1, j \ge 2 \quad \rightarrow (12)$$

$$-Q_{jn}^{'}(x) = -\lambda Q_{jn}(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda g_k, \quad n \ge a, j \ge 2 \quad \rightarrow (13)$$

$$-R_n^{'}(x) = -\lambda R_n(x) + \sum_{k=1}^{n} Q_{j,n}(x)r(x) + \sum_{k=1}^{n-N} R_{n-k}(x)\lambda g_k, \quad n \ge N \quad \rightarrow (14)$$

Taking LT on both sides of the equations (1)-(14), we get

$$\theta\tilde{P}_{i0}(\theta) - P_{io}(0) = \lambda\tilde{P}_{i0}(\theta) - \sum_{m=d}^{b} P_{mi}(0)\tilde{S}(\theta) - \sum_{l=1}^{\infty} Q_{li}(0)\tilde{S}(\theta) - \sum_{k=1}^{i-a} P_{i-k,0}(\theta)\lambda g_k, \quad a \le i \le d \quad \rightarrow (15)$$

$$\theta\tilde{P}_{i0}(\theta) - P_{i0}(0) = \lambda\tilde{P}_{i0}(\theta) - \sum_{m=d}^{b} P_{mi}(0)\tilde{S}(\theta) - \sum_{l=1}^{\infty} Q_{li}(0)\tilde{S}(\theta), \quad d+1 \le i \le b \quad \rightarrow (16)$$

$$\theta\tilde{P}_{dn}(\theta) - P_{dn}(0) = \lambda\tilde{P}_{dn}(\theta) - \sum_{k=1}^{n} \tilde{P}_{d,n-k}(\theta)\lambda g_k - \sum_{k=1}^{d-a} \tilde{P}_{d-k,0}(\theta)\lambda g_{k+n}, \quad n \ge 1 \quad \rightarrow (17)$$

$$\theta\tilde{P}_{in}(\theta) - P_{in}(0) = \lambda\tilde{P}_{in}(\theta) - \sum_{k=1}^{n} \tilde{P}_{i,n-k}(\theta)\lambda g_k, \quad d < i < b-1, n \ge 1 \quad \rightarrow (18)$$

$$\theta\tilde{P}_{bn}(\theta) - P_{bn} = \lambda\tilde{P}_{bn}(\theta) - \sum_{m=d}^{b} P_{m,b+n}(0)\tilde{S}(\theta) - \sum_{l=1}^{\infty} Q_{l,b+n}(0)\tilde{S}(\theta) - \sum_{k=1}^{n} \tilde{P}_{b,n-k}(\theta)\lambda g_k, \quad n \ge 1 \quad \rightarrow (19)$$

$$\theta\tilde{C}_0(\theta) - C_0(0) = \lambda\tilde{C}_0(\theta) - \sum_{m=a}^{b} P_{m0}(0)\tilde{C}(\theta) \quad \rightarrow (20)$$

$$\theta\tilde{C}_n(\theta) - C_n(0) = \lambda\tilde{C}_n(\theta) - \sum_{m=d}^{b} P_{mn}(0)\tilde{C}(\theta) - \sum_{k=1}^{n} \tilde{C}_{n-k}(\theta)\lambda g_k, \quad 1 \le n \le a-1 \quad \rightarrow (21)$$

$$\theta\tilde{C}_n(\theta) - C_n(0) = \lambda\tilde{C}_n(\theta) - \sum_{k=1}^{n} \tilde{C}_{n-k}(\theta)\lambda g_k, \quad n \ge a \quad \rightarrow (22)$$

$$\theta\tilde{Q}_{10}(\theta) - Q_{10}(0) = \lambda\tilde{Q}_{10}(\theta) - C_0(0)\tilde{V}(\theta), \quad \rightarrow (23)$$

$$\theta\tilde{Q}_{1n}(\theta) - Q_{1n}(0) = \lambda\tilde{Q}_{1n}(\theta) - C_n(0)\tilde{V}(\theta) - \sum_{k=1}^{n} \tilde{Q}_{1,n-k}(\theta)\lambda g_k, \quad n \ge 1 \quad \rightarrow (24)$$

$$\theta\tilde{Q}_{j0}(\theta) - Q_{j0}(0) = \lambda\tilde{Q}_{j0}(\theta) - Q_{j-1,0}(0)\tilde{V}(\theta), \quad j \ge 2 \quad \rightarrow (25)$$

$$\theta\tilde{Q}_{jn}(\theta) - Q_{jn}(0) = \lambda\tilde{Q}_{jn}(\theta) - Q_{j-1,n}(0)\tilde{V}(\theta) - \sum_{k=1}^{n} \tilde{Q}_{j,n-k}(\theta)\lambda g_k, \quad 1 \le n \le a-1, j \ge 2 \rightarrow (26)$$

$$\theta\tilde{Q}_{jn}(\theta) - Q_{jn}(0) = \lambda\tilde{Q}_{jn}(\theta) - \sum_{k=1}^{n} \tilde{Q}_{j,n-k}(\theta)\lambda g_k, \quad n \ge a, j \ge 2 \quad \rightarrow (27)$$

$$\theta \tilde{R}_n(\theta) - R_n(0) = \lambda \tilde{R}_n(\theta) - \sum_{l=1}^{n} Q_{l,n}(0)\tilde{R}(\theta) - \sum_{k=1}^{n-N} R_{n-k}(\theta)\lambda g_k, \quad n \ge N \qquad \to (28)$$

4. Queue Size Distribution

To obtain the probability generating function of the queue size at an arbitrary time epoch, the following probability generating functions are defined.

$$\begin{cases} \tilde{P}_i(z,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{ij}(\theta) z^{ij} \\ P_i(z,0) = \sum_{j=0}^{\infty} P_{ij}(0) z^j, \quad d \le i \le b \\ \tilde{Q}_j(z,\theta) = \sum_{n=0}^{\infty} \tilde{Q}_{jn}(\theta) z^n \\ Q_j(z,0) = \sum_{n=0}^{\theta} Q_{jn}(0) z^n, \quad j \ge 1 \qquad \to (29) \\ \tilde{C}(z,\theta) = \sum_{n=0}^{\infty} \tilde{C}_n(\theta) z^n \\ C(z,0) = \sum_{n=0}^{\infty} C_n(0) z^n \\ \tilde{R}(z,\theta) = \sum_{n=N}^{\infty} \tilde{R}_n(\theta) z^n \\ R(z,0) = \sum_{n=N}^{\infty} R_n(0) z^n \end{cases}$$

Multiplying the equation (23) by z^0 and (24) by z^n ($n \ge 1$) taking summation from n=0 to ∞ and using (29), we get

$$(\theta - \lambda + \lambda X(z))\tilde{Q}_1(z,\theta) = Q_1(z,0) - \tilde{V}(\theta) C(z,0) \qquad \to (30)$$

Multiplying the equation (25) by z^0 , (26) by z^n ($1 \le n \le a-1$) and (27) by z^n ($n \ge a$) taking summation from n=0 to ∞ and using (29), we get

$$(\theta - \lambda + \lambda X(z))\tilde{Q}_j(z,\theta) = Q_j(z,0) - \tilde{V}(\theta)\sum_{n=0}^{a-} Q_{j-1,n}(0) z^n \qquad \to (31)$$

Multiplying the equation (20) by z^0 , (21) by z^n ($1 \le n \le a-1$) and (22) by z^n ($n \ge a$) taking summation from n=0 to ∞ and using (29), we get

$$(\theta - \lambda + \lambda X(z))\tilde{C}(z,\theta) = C(z,0) - \tilde{C}(\theta)\left[\sum_{m=a}^{b} P_{m0}(0) z^n + \sum_{n=1}^{a-1}\sum_{m=d}^{b} P_{mn}(0) z^n\right] \qquad \to (32)$$

Multiplying the equation (15) by z^0 with i=d, (17) by z^n ($n \ge 1$) and taking summation from n=0 to ∞ and using (29), we get

$$(\theta - \lambda + \lambda X(z))\tilde{P}_d(z,\theta) = P_d(z,0) - \tilde{S}(\theta)\left[\sum_{m=d}^{b} P_{md}(0) + \sum_{l=1}^{\infty} Q_{ld}(0)\right] - \frac{\lambda}{z^d}\sum_{i=a}^{d-1} \tilde{P}_{i0}(\theta) z^i * \left[X(z) - G_i(z)\right] \qquad \to (33)$$

where $G_i(z) = \sum_{k=}^{d-1-i} g_k z^k$

Multiplying the equation (16) by z^0, (18) by z^n ($n \ge 1$) and taking summation from n=0 to ∞ and using (29), we get

$$(\theta - \lambda + \lambda X(z))\tilde{P}_i(z,\theta) = P_i(z,0) - \tilde{S}(\theta)\left[\sum_{m=d}^{b} P_{mi}(0) + \sum_{l=1}^{\infty} Q_{li}(0)\right] \quad d+1 \le i \le b-1 \qquad \to (34)$$

Multiplying the equation (16) by z^0 with i=b, (19) by z^n ($n \ge 1$) and taking summation from n=0 to ∞ and using (29), we get

$$(\theta - \lambda + \lambda X(z))\tilde{P}_b(z,\theta) = P_b(z,0) - \frac{\tilde{S}(\theta)}{z^b}\left[\sum_{m=d}^{b}\left(P_m(z,0) - \sum_{n=0}^{b-1} P_{mn}(0) z^n\right) + \sum_{l=1}^{\infty}\left(Q_l(z,0) - \sum_{n=0}^{b-1} Q_{ln}(0) z^n\right)\right] \qquad \to (35)$$

Multiplying the equation (28) by z^0 ($n \ge 1$) and taking summation from n=0 to ∞ and using (29), we get

$$(\theta - \lambda + \lambda X(z))\tilde{R}(z,\theta) = R(z,0) - \tilde{R}(\theta)\left[\sum_{l=1}^{b}\left(Q_l(z,0) - \sum_{n=0}^{N-} Q_{ln}(0) z^n\right)\right] \qquad \to (36)$$

Substituting $\theta = \lambda - \lambda X(z)$ in (30) through (36), we get

$$Q_1(z,0) = \tilde{V}(\lambda - \lambda X(z)) C(z,0) \qquad \to (37)$$

$$\tilde{Q}_j(z,0)=\tilde{V}(\lambda-\lambda X(z))\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n \qquad \rightarrow(38)$$

$$C(z,0)=\tilde{C}(\lambda-\lambda X(z))\begin{bmatrix}\sum_{m=a}^{d-1}P_{m0}(0)z^0+\\ \sum_{n=0}^{a-1}\sum_{m=d}^{b}P_{mn}(0)z^n\end{bmatrix} \qquad \rightarrow(39)$$

$$P_d(z,0)=\tilde{S}(\lambda-\lambda X(z))\left[\sum_{m=d}^{b}P_{md}(0)+\sum_{l=1}^{\infty}Q_{ld}(0)\right]$$
$$+\frac{\lambda}{z^d}\sum_{i=a}^{d-}\tilde{P}_{i0}(\lambda-\lambda X(z))z^i*[X(z)-G_i(z)]\rightarrow(40)$$

where $G_i(z)=\sum_{k=}^{d-1-i}g_k z^k$

$$P_i(z,0)=\tilde{S}(\lambda-\lambda X(z))\left[\sum_{m=d}^{b}P_{mi}(0)+\sum_{l=1}^{\infty}Q_{li}(0)\right]$$
$$d+1\le i\le b-1 \qquad \rightarrow(41)$$

$$P_b(z,0)=\left[\tilde{S}(\lambda-\lambda X(z))\begin{pmatrix}\sum_{m=d}^{b-1}P_m(z,0)-\sum_{m=d}^{b}\sum_{n=0}^{b-1}P_{mn}(0)z^n\\ +\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{l=1}^{\infty}\sum_{n=0}^{b-1}Q_{ln}(0)z^n\end{pmatrix}\right]$$
$$*\frac{1}{z^b-\tilde{S}(\lambda-\lambda X(z))} \qquad \rightarrow(42)$$

$$R(z,0)=\tilde{R}(\lambda-\lambda X(z))\left[\sum_{l=1}^{\infty}(Q_l(z,0)-\sum_{n=0}^{N-1}Q_{ln}(0)z^n\right]\rightarrow(43)$$

From the equations (30) and (37), we get

$$\tilde{Q}_1(z,\theta)=\frac{(\tilde{V}(\lambda-\lambda X(z))-\tilde{V}(\theta))C(z,0)}{\theta-\lambda+\lambda X(z)} \qquad \rightarrow(44)$$

From the equations (31) and (38), we get

$$\tilde{Q}_j(z,\theta)=\frac{\begin{bmatrix}(\tilde{V}(\lambda-\lambda X(z))-\tilde{V}(\theta))\\ \sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n\end{bmatrix}}{\theta-\lambda+\lambda X(z)},\quad j\ge 2 \qquad \rightarrow(45)$$

From the equations (32) and (39), we get

$$\tilde{C}(z,\theta)=\left(\tilde{C}(\lambda-\lambda X(z))-\tilde{C}(z,\theta)\right)$$
$$\begin{bmatrix}\sum_{m=a}^{d-1}P_{m0}(0)z^0+\\ \sum_{n=0}^{a-1}\sum_{m=d}^{b}P_{mn}(0)z^n\end{bmatrix}*\frac{1}{\theta-\lambda+\lambda X(z)} \qquad \rightarrow(46)$$

From the equations (33) and (40), we get

$$\tilde{P}_d(z,\theta)=\begin{bmatrix}(\tilde{S}(\lambda-\lambda X(z))-\tilde{S}(\theta))\\ \left[\sum_{m=d}^{b}P_{md}(0)+\sum_{l=1}^{\infty}Q_{ld}(0)\right]\\ +\frac{\lambda}{z^d}\sum_{i=a}^{d-}\tilde{P}_{i0}(\lambda-\lambda X(z))z^i\\ *[X(z)-G_i(z)]\end{bmatrix}$$
$$*\frac{1}{\theta-\lambda+\lambda X(z)} \qquad \rightarrow(47)$$

From the equations (34) and (41), we get

$$\tilde{P}_i(z,\theta)=\frac{(\tilde{S}(\lambda-\lambda X(z))-\tilde{S}(\theta))\left[\sum_{m=d}^{b}P_{mi}(0)+\sum_{l=1}^{\infty}Q_{li}(0)\right]}{\theta-\lambda+\lambda X(z)}$$
$$d+1\le i\le b-1 \qquad \rightarrow(48)$$

From the equations (35) and (42), we get

$$P_b(z,0)=\begin{bmatrix}(\tilde{S}(\lambda-\lambda X(z))-\tilde{S}(\theta)) \\ \left(\begin{array}{l}\sum_{m=d}^{b-1}P_m(z,0)-\sum_{m=d}^{b}\sum_{n=0}^{b-1}P_{mn}(0)z^n \\ +\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{l=1}^{\infty}\sum_{n=0}^{b-1}Q_{ln}(0)z^n\end{array}\right)\end{bmatrix} \frac{1}{z^b-\tilde{S}(\lambda-\lambda X(z))(\theta-\lambda+\lambda X(z))} \rightarrow (49)$$

From the equations (36) and (43), we get

$$\tilde{R}(z,\theta)=\frac{\begin{bmatrix}[\tilde{R}(\lambda-\lambda X(z))-\tilde{R}(\theta)] \\ \left[\sum_{l=1}^{\infty}(Q_l(z,0)-\sum_{n=0}^{N-1}Q_{ln}(0)z^n\right]\end{bmatrix}}{(\theta-\lambda+\lambda X(z))} \rightarrow (50)$$

Substituting θ=0, in equations (44) through (50), we get

$$\tilde{Q}_1(z,0)=\frac{(\tilde{V}(\lambda-\lambda X(z))-1)C(z,0)}{-\lambda+\lambda X(z)} \rightarrow (51)$$

$$\tilde{Q}_j(z,0)=\frac{\begin{bmatrix}(\tilde{V}(\lambda-\lambda X(z))-1) \\ \sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n\end{bmatrix}}{-\lambda+\lambda X(z)},\quad j\geq 2 \rightarrow (52)$$

$$\tilde{C}(z,0)=\left(\tilde{C}(\lambda-\lambda X(z))-1\right)\begin{bmatrix}\sum_{m=a}^{d-1}P_{m0}(0)z^0+ \\ \sum_{n=0}^{a-1}\sum_{m=d}^{b}P_{mn}(0)z^n\end{bmatrix} * \frac{1}{-\lambda+\lambda X(z)} \rightarrow (53)$$

$$\tilde{P}_d(z,0)=\begin{bmatrix}(\tilde{S}(\lambda-\lambda X(z))-1) \\ \left[\sum_{m=d}^{b}P_{md}(0)+\sum_{l=1}^{\infty}Q_{ld}(0)\right] \\ +\frac{\lambda}{z^d}\sum_{i=a}^{d-}\tilde{P}_{i0}(\lambda-\lambda X(z))z^i \\ *[X(z)-G_i(z)]\end{bmatrix} * \frac{1}{-\lambda+\lambda X(z)} \rightarrow (54)$$

$$\tilde{P}_i(z,0)=\frac{(\tilde{S}(\lambda-\lambda X(z))-1)\left[\sum_{m=d}^{b}P_{mi}(0)+\sum_{l=1}^{\infty}Q_{li}(0)\right]}{-\lambda+\lambda X(z)}$$
$$d+1\leq i\leq b-1 \rightarrow (55)$$

$$P_b(z,0)=\begin{bmatrix}(\tilde{S}(\lambda-\lambda X(z))-1) \\ \left(\begin{array}{l}\sum_{m=d}^{b-1}P_m(z,0)-\sum_{m=d}^{b}\sum_{n=0}^{b-1}P_{mn}(0)z^n \\ +\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{l=1}^{\infty}\sum_{n=0}^{b-1}Q_{ln}(0)z^n\end{array}\right)\end{bmatrix} * \frac{1}{z^b-\tilde{S}(\lambda-\lambda X(z))(-\lambda+\lambda X(z))} \rightarrow (56)$$

$$\tilde{R}(z,0)=\frac{\begin{bmatrix}[\tilde{R}(\lambda-\lambda X(z))-1] \\ \left[\sum_{l=1}^{\infty}(Q_l(z,0)-\sum_{n=0}^{N-1}Q_{ln}(0)z^n\right]\end{bmatrix}}{(-\lambda+\lambda X(z))} \rightarrow (57)$$

Let P(z) be the probability generating function of the queue size at an arbitrary time epoch. Then,

$$P(z)=\sum_{i=a}^{d-1}\tilde{P}_i(0)+\tilde{P}_d(z,0)+\sum_{i=d+1}^{b-1}\tilde{P}_i(z,0)+\tilde{P}_b(z,0)$$
$$+\tilde{C}(z,0)+\sum_{j=1}^{\infty}\tilde{Q}_j(z,0)+\tilde{R}(z,0) \rightarrow (58)$$

Using the equation (51) through (57) in (58), we get

$$
\begin{aligned}
P(z) = \sum_{i=a}^{d-1} \tilde{P}_{i0}(0) &+ \frac{\left[\begin{array}{l}(\tilde{S}(\lambda-\lambda X(z))-1) \\ \left[\sum_{m=d}^{b} P_{md}(0)+\sum_{l=1}^{\infty} Q_{ld}(0)\right] \\ +\frac{\lambda}{z^{d}} \sum_{i=a}^{d-} \tilde{P}_{i0}(\lambda-\lambda X(z)) z^{i} \\ *[X(z)-G_{i}(z)]\end{array}\right]}{-\lambda+\lambda X(z)} \\
&+ \sum_{i=d+1}^{b-1} \frac{(\tilde{S}(\lambda-\lambda X(z))-1)\left[\sum_{m=d}^{b} P_{mi}(0)+\sum_{l=1}^{\infty} Q_{li}(0)\right]}{-\lambda+\lambda X(z)} \\
&+ \frac{\left[\begin{array}{l}(\tilde{S}(\lambda-\lambda X(z))-1) \\ \left(\begin{array}{l}\sum_{m=d}^{b-1} P_{m}(z,0)-\sum_{m=d}^{b} \sum_{n=0}^{b-1} P_{mn}(0) z^{n} \\ +\sum_{l=1}^{\infty} Q_{l}(z,0)-\sum_{l=1}^{\infty} \sum_{n=0}^{b-1} Q_{ln}(0) z^{n}\end{array}\right)\end{array}\right]}{z^{b}-\tilde{S}(\lambda-\lambda X(z))(-\lambda+\lambda X(z))} \\
&+ \frac{(\tilde{C}(\lambda-\lambda X(z))-1)\left[\begin{array}{l}\sum_{m=a}^{d-1} P_{m0}(0) z^{0}+ \\ \sum_{n=0}^{a-1} \sum_{m=d}^{b} P_{mn}(0) z^{n}\end{array}\right]}{-\lambda+\lambda X(z)} \\
&+ \frac{(\tilde{V}(\lambda-\lambda X(z))-1) C(z,0)}{-\lambda+\lambda X(z)} \\
&+ \frac{\left[\begin{array}{l}(\tilde{V}(\lambda-\lambda X(z))-1) \\ \qquad \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^{n}\end{array}\right]}{-\lambda+\lambda X(z)} \\
&+ \frac{\left[\begin{array}{l}[\tilde{R}(\lambda-\lambda X(z))-1] \\ \left[\sum_{l=1}^{\infty}(Q_{l}(z,0)-\sum_{n=0}^{N-1} Q_{ln}(0) z^{n}\right]\end{array}\right]}{(-\lambda+\lambda X(z))} \quad \rightarrow (59)
\end{aligned}
$$

Let

$$
\left\{
\begin{aligned}
& p_i = \sum_{m=d}^{b} P_{mi}(0), \\
& q_i = \sum_{l=1}^{\infty} Q_{li}(0), \\
& c_i = p_i + q_i, \\
& \psi(\tilde{S}, z) = \sum_{i=a}^{d-1} \tilde{P}_{i0}(0), \\
& \phi(\tilde{S}, z) = \sum_{i=a}^{d-1} (\tilde{P}_{i0}(\lambda-\lambda X(z)) - \tilde{P}_{i0}(0)) z^{i-d} (X(z)-G_i(z)) \\
& f(\tilde{S}, z) = \tilde{S}(\lambda-\lambda X(z)) c_d \\
& \qquad + \frac{\lambda}{z^d} \sum_{i=a}^{d-1} \tilde{P}_{i0}(\lambda-\lambda X(z)) z^i (X(z)-G_i(z)) \quad \rightarrow (60) \\
& \qquad + \sum_{m=d+1}^{b-1} \tilde{S}(\lambda-\lambda X(z)) c_m \\
& \qquad + \tilde{V}(\lambda-\lambda X(z)) \tilde{C}(\lambda-\lambda X(z)) \sum_{n=0}^{a-1} P_n z^n \\
& \qquad + \tilde{V}(\lambda-\lambda X(z)) \sum_{n=0}^{a-1} q_n z^n - \sum_{n=0}^{b-1} c_n z^n \\
& u(z,0) = (z^b - \tilde{S}(\lambda-\lambda X(z))) R(z,0)
\end{aligned}
\right.
$$

Using the equation (60) and (59) is simplified as

$$
P(z) = \frac{\left[\begin{array}{l}(z^{b}-\tilde{S}(\lambda-\lambda X(z)))(-\lambda+\lambda X(z)) \psi(\tilde{S}, z) \\ +\lambda(Z^{b}-\tilde{S}(\lambda-\lambda X(z))) \phi(\tilde{S}, z) \\ +(\tilde{S}(\lambda-\lambda X(z))-1) f(\tilde{S}, z)+(\tilde{V}(\lambda-\lambda X(z)) \\ \tilde{C}(\lambda-\lambda X(z))-1)(\sum_{n=0}^{a-1} P_{n} z^{n}) \\ (z^{b}-\tilde{S}(\lambda-\lambda X(z)))+(\tilde{V}(\lambda-\lambda X(z))-1) \\ (z^{b}-\tilde{S}(\lambda-\lambda X(z))) \sum_{n=0}^{a-1} q_{n} z^{n}\end{array}\right]}{(-\lambda+\lambda X(z))(z^{b}-\tilde{S}(\lambda-\lambda X(z)))} + U(z,0) \quad \rightarrow (61)
$$

In P(z) functions $\psi(\tilde{S}, z)$, $\phi(\tilde{S}, z)$ and $f(\tilde{S}, z)$ which involve LST of the unknown functions $\tilde{P}_{i0}(0)$ are present. While modelling a non- accessible batch service queue, such complexity will not occur. In order to resolve this complexity, $\tilde{P}_{i0}(\theta)$ are expressed in terms of known function $\tilde{S}(\theta)$.

4. Expected length of Idle Period.

Expected Idle Period can be defined as the mean time gap between the completion epoch of a service and initiation epoch of the next service, inclusive of multiple vacations and closedown period. Let I be the random variable idle period. The expected length of idle period is given by $E(I)=E(I_1)+E(C)$ where I_1 is the random variable denoting idle period due to multiple vacation process and E(C) is the expected closedown time.

Let U be the random variable defined by

$$U = \begin{cases} 0, & \text{if the server finds at least 'a' customers after the first vacation} \\ 1, & \text{if the server finds less that 'a' customers after the first vacation} \end{cases}$$

then, using conditional expectation, the expected length of idle period $E(I_1)$ is given by

$$E(I_1) = E(I_1/U=0)P(U=0)+ E(I_1/U=1)P(U=1)$$

$$= E(V)P(U=0)+(E(V)+E(I_1))P(U=1)$$

Where E(V) is the mean vacation time. Solving for $E(I_1)$, we get

$$E(I_1) = E(V)/[1-P(U=1)] = E(V)/[P(U=0)] \quad \rightarrow (62)$$

To find P(U=0), we do some algebra in the equations (29) and (37) we get

$$Q_1(z,0) = \sum_{n=0}^{\infty} Q_{1n}(0) z^n$$

$$= \tilde{V}(\lambda - \lambda X(Z))\left[\tilde{C}(\lambda - \lambda X(z))\sum_{n=0}^{a-1} p_n z^n\right]$$

$$= \left(\sum_{n=0}^{\infty} \alpha_n z^n\right)\left[\sum_{j=0}^{\infty} \beta_j z^j \sum_{n=0}^{a-1} p_n z^n\right]$$

equating the coefficient of z^n (n=0,1,2,....,a-1) on both sides, we get

$$Q_{1n}(0) = \sum_{i=0}^{n}\left(\sum_{j=0}^{n-1} \alpha_j \beta_{n-i-j}\right) p_n$$

$$P(U=0) = 1 - \sum_{n=0}^{a-1} Q_{1n}(0)$$

$$= 1 - \sum_{n=0}^{a-1}\sum_{i=0}^{n}\left(\sum_{j=0}^{n-1} \alpha_j \beta_{n-i-j}\right) p_n \quad \rightarrow (63)$$

where α_i, β_i are the probabilities of 'i' customers arrive during vacation and closedown time. Using (62) and (63), the expected idle period E(I) is obtained as

$$E(I) = \frac{E(V)}{1 - \sum_{n=0}^{a-1}\sum_{i=0}^{n}\left(\sum_{j=0}^{n} \alpha_j \beta_{n-i-j}\right)} + E(C)$$
$$+ E(R) \rightarrow (64)$$

5. Expected length of Busy Period

In this section, the expected length of busy period which is useful to find the overall cost of the system is derived. Using the conditional expectation concept, the expected length of busy period is derived as follows:

Busy period is defined as the time interval from the moment when the server starts serving the queue, after returning from a vacation until the server leaves the system for another vacation.

Let B be the random variable for 'busy period'. Define another random variable J as

J=0, if the server finds less than 'a' customers in the queue
after first service
1, if the server finds atleast 'a' customers in the queue
after first service

Now the expected length of busy period E(B) is

$$E(B) = E(B/J=0)P(J=0)+E(B/J=1)P(J=1)$$
$$= E(B/J=0)P(J=0)+(E(B)+E(S))P(J=1)$$
$$= E(B/J=0)P(J=0)+(E(B)+E(S))(1-P(J=0))$$

Where E(S) is expected service time. Solving for E(B),

$$E(B) = E(S)/P(J=0)$$

$$=E(S)/\sum_{i=0}^{a-1} P_i \quad \rightarrow (65)$$

is obtained.

6. Expected Queue Length

The expected queue length E(Q) (i.e., average number of customers waiting in the queue) at an arbitrary time epoch is obtained by differentiating p(z) at z=1 and is given by E(Q)= $\sum_{n=0}^{\infty} np_n = p^1(1)$. From the equation (61) using L'Hospital's rules and evaluating the limit, $\lim_{z\to 1} \frac{d_p(z)}{dz}$, we get

$$E(Q) = \frac{\varphi''}{4X_1} + \frac{[k1 - 12 * k2]}{8\lambda^2 X_1{}^2 (b - S_1)^2} + \frac{[(\lambda X_1)k3 - k4]}{2(\lambda^2 X_1{}^2)}$$
$$+ \frac{[\lambda X_1 (2V_1 \sum_{n=0}^{a-1} nq_n + V_2 \sum_{n=0}^{a-1} q_n) - (\lambda X_2)(V_1 \sum_{n=0}^{a-1} q_n)]}{2(\lambda^2 X_1{}^2)} \rightarrow (65)$$

where

$k1 = \{4\lambda X_1(b-S_1)\}\{2S_2.f'+3S_1f''\}$
$k2 = (S_1.f')[(\lambda X_1)(b(b-1)-S_2)+(\lambda C_1X_2)(b-S_1)]$
$k3=\{(2V_1C_1+V_2+C_2)(\sum_{n=0}^{a-1} p_n)+(V_1+C_1)(\sum_{n=0}^{a-1} np_n)+(V_1+C_1)(\sum_{n=0}^{a-1} p_n)\}$
$k4=\{(V_1+C_1)(\sum_{n=0}^{a-1} p_n)(\lambda X_2)\}$
$S_1=\lambda X_1E(S)$
$S_2= \lambda X_2E(S)+ \lambda^2X^2{}_1E(S^2)$
$X_1=E(X)$
$X_2=X'''(1)$
$V_1= \lambda X_1E(V)$
$V_2= \lambda X_2E(V)+ \lambda^2X^2{}_1E(V^2)$
$C_1= \lambda X_1E(C)$
$C_2= \lambda X_2E(C)+\lambda^2X^2{}_1E(C^2)$

7. Expected waiting time

The expected waiting time is obtained by using the Little's formula as E(W)=E(Q)/λE(X), where E(Q) is expected queue length as in (65).

8. Cost Model

Cost Analysis is an important phenomenon in any system. In this section, the total average coast of the queueing system is derived with the following assumptions.

C_s : Start up cost per cycle.
C_h : Holding cost per customer per unit time
C_0 : Operating cost per unit time
C_r : Reward due to vacation per unit time
C_u : Closedown cost per unit time.

The length of cycle is the sum of the idle period and busy period. From the equations(63) and (64), the expected length of cycle, $E(T_c)$ is obtained as

$$E(T_c) = E(I) + E(B)$$
$$= \frac{E(V)}{1 - \sum_{n=0}^{a-1}\sum_{i=0}^{n}\left(\sum_{j=0}^{n} \alpha_j \beta_{n-i-j}\right) p_n} + E(C) + E(R) + \frac{E(S)}{\sum_{n=0}^{a-1} p_i} \rightarrow (66)$$

The total average cost per unit is given by
Total average cost = start up cost per cycle
+ holding cost of number of customers in the queue
+ operating cost*ρ
+ closedown time cost
+ setup cost per cycle
- reward due to vacation per unit Time

$$Total\ average\ \cos t = \frac{\left[\begin{array}{l} C_s - C_r \dfrac{E(V)}{P(U=0)} \\ + C_u E(C) + C_r E(R) \end{array}\right]}{E(T_c)}$$

$$+ C_h E(Q) + C_0 \rho \qquad \rightarrow (67)$$

Where $\rho = \lambda E(X)E(S)/b$ and $E(T_c)$,$E(Q)$ are given in (66) and (65) respectively

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