String Cosmological Model in Axially Symmetric Bianchi-I Space Time With and Without Magnetic Field

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- *Abstract* - The present paper provides a string cosmological solution in axially symmetric Bianchi – I space time with and without magnetic field using different assumption. To get a deterministic model we have assumed the condition $\lambda = l + m\upsilon$ (Where l and m are constant). under the different cases m = 0, l = m = 0 and l = 0 (for magnetic field) & K = 0 (for absence of magnetic field). Various physical parameters of the model have been also evaluated.

- KEYWORD :-

Cosmological model, axially symmetric Bianchi – I space time electromagnetic field tensor, magnetic field, shear strain, expansion scalar.

1. INTRODUCTION :-

Various researchers in theory of relativity have focussed their mind towards the study of string cosmology. Takabayashi (9), Letelier (6), Stachel (8), Banerjee et al.(1) Cohen and Kaplan (5) have found some cosmological solutions in Bianchi – I space time following the technique used by Letelier and satchel with the magnetic field. Bali and Anjali (2) have found Bianchi – I magnetized string dust cosmological model. Melvin (7) in his solution for dust and de-electromagnetic field argued that the presence of magnetic field is not unrealistic as it appears to be, because for a large part of the history of evolution matter was highly ionized and matter and field were smoothly coupled. Later during cooling as a result of expansion the ions combined to form neutral matter. Some other worker in this line are Baysal et al.(4), Bali and Pareek (3), Yadav et al..(10).

In this paper we have studied the string cosmology in axially symmetric Bianchi – I space time with and without magnetic field using different assumptions. Various physical parameters of the model have been also evaluated.

2. THE FIELD EQUATIONS :-

Here we take an axially symmetric Bianchi – I metric given by –

(2.1) $\quad ds^2 = -dt^2 + e^{2\lambda} . dx^2 + e^{2\upsilon} . (dy^2 + dz^2)$

Where λ and υ are function of t only.

Now, the energy – momentum tensor for the string dust with a magnetic field along the direction of the string, i,e the x direction is given by ---

(2.2) $\quad T_\mu^\delta + E_\mu^\delta = \rho\, u_\mu u^\delta - \mathcal{T} x_\mu x^\delta +$

$$\frac{1}{4\pi}\left[F_\mu^i F_i^\delta - \frac{1}{4} F_{ij} F^{ij} \zeta_\mu^\delta \right]$$

Where T_μ^δ is the stress – energy tensor for a string dust system, E_μ^δ is that for magnetic field & F_{ij} is the electromagnetic field tensor. The other terms have already been explained in the previous system. In the co-moving co-ordinates system

$$u_\mu = \zeta_4^\delta \quad \text{and}$$

(2.3) $\quad T_4^4 = -\rho\,, \ T_1^1 = -\mathcal{T}, \ T_2^2 = T_3^3 = 0 = T_\mu^\delta \quad \text{(for } \mu \neq \delta \text{)}$

Again since the magnetic field is being assumed in the X-direction, F_{23} is the only non-zero component of the electromagnetic field tensor.

Maxwell Equation

$$F_{[\mu\delta,\alpha]} = 0 \ \text{ and } \ \left[F^{\mu\delta} (-g)^{1/2} \right]_\mu = 0$$

Now lead to the result.

(2.4) $\quad - F_{23} = K$

where K is constant. Therefore the components of stress energy tensor for the electromagnetic field are

(2.5) $\quad E_4^4 = E_1^1 = -E_2^2 = -E_3^3$

$$= \frac{-K^2}{8\pi} \ \exp\ (-4\upsilon)$$

Now choosing unit such that $8\pi G = 1$, the surviving components of Einstein field Equations given by -

$$(2.6) \quad R_\mu^\delta \ - \ \frac{1}{2} \ \zeta_\mu^\delta . R \ = - \left(T_\mu^\delta \ + \ E_\mu^\delta \right)$$

are

$$(2.7) \quad 2\dot{\lambda}\dot{\upsilon} + \dot{\upsilon}^2 = \rho + \frac{K^2}{8\pi} \ . \exp\ (-4\upsilon)$$

$$(2.8) \quad 2\ddot{\upsilon} + 3\dot{\upsilon}^2 = \text{Ʈ} + \frac{K^2}{8\pi} . \exp\ (-4\upsilon)$$

$$(2.9) \quad \ddot{\lambda} + \dot{\lambda}^2 + \dot{\upsilon}^2 + \dot{\lambda}\dot{\upsilon} \ = \ - \ \frac{K^2}{8\pi} . \exp\ (-4\upsilon)$$

where the dot denotes the differentiation with respect to time t.

The proper volume R^3 ; expansion scalar (θ) and shear scalar σ^2 are respectively given by –

$$(2.10) \quad R^3 = \exp\ (\lambda + 2\upsilon)$$

$$(2.11) \quad \theta = U_{;\mu}^\mu = \dot{\lambda} + 2\dot{\upsilon} = \frac{3\dot{R}}{R}$$

$$(2.12) \quad \sigma^2 = \sigma_{\mu\delta}\sigma^{\mu\delta} = \dot{\lambda}^2 + 2\dot{\upsilon}^2 - \frac{1}{3}\theta^2$$

where,

$$(2.13) \sigma_{\mu\delta} = \frac{1}{2} [u_{\mu\delta+} u_{\delta\mu+} u_\mu u^i u_{\delta i+} u_\delta u^i u_{\mu i}] - \frac{1}{3}\theta \ (g_{\mu\delta} + u_\mu u_\delta).$$

3. SOLUTION OF THE FIELD EQUATION :--

We have three equation (2.7) to (2.9) in four unknown λ, υ ρ and Ʈ . Therefore the system is indeterminate. To make the system determinate, we required one more relation. For this we choose,

$$(3.1) \quad \lambda = 1 + m\upsilon$$

(where l and m are constant)

Case- I

If we put m = 0 Then

$$\lambda = 1$$

$$\dot{\lambda} = 0$$

$$\ddot{\lambda} = 0$$

In this case (2.9) reduces to

$$(3.2) .... \ddot{\upsilon} + \dot{\upsilon}^2 = - \frac{K^2}{8\pi} \ \exp\ (-4\ \upsilon)$$

This equation can be written as an integral equation :-

$$(3.3) \qquad \int d(\dot{\upsilon}^2 e^{2\upsilon}) =$$

$$- \frac{K^2}{4\pi} \ e^{-2\upsilon} \ d\upsilon \ + \ K_1$$

Where K_1 is the constant of integration so we get,

$$(3.4) \qquad \dot{\upsilon}^2 = K_1 \ . \ e^{-2\upsilon} \ + \ \frac{K^2}{8\pi} \ e^{-4\upsilon}$$

This can be again written as an integral form as –

$$(3.5) \qquad \int \frac{e^{2\upsilon}}{\left\{ K_1 \ e^{2\upsilon} + \frac{K^2}{8\pi} \right\}^{1/2}} \ = \ \pm \ (t - t_0)$$

where t_0 is another integration constant. Integrating (3.5) we get

$$(3.6) \qquad e^{2\upsilon} \ = \ K_1 \left(t - t_0 \right)^2 - \frac{K^2}{8\pi K_1}$$

ρ and Ʈ can now be found from the equation (2.7) and (2.8) respectively as :-

$$(3.7) \quad \rho = K_1 \left[K_1(t - t_0)^2 - \frac{K^2}{8\pi K_1} \right]^{-1}$$

$$(3.8) \quad \text{Ʈ} = \left[K_1^2(t - t_0)^2 - \frac{3K^2}{8\pi} \right] \left[K_1(t - t_0)^2 - \frac{K^2}{8\pi K_1} \right]^{-2}$$

(3.9) ρ_p = $\rho - \mathcal{T}$ = $\dfrac{K^2}{4\pi}$

$$\left[K_1(t - t_0)^2 - \dfrac{K^2}{8\pi K_1} \right]^{-2}$$

Case II

Here we put $l = m = 0$ in equation (3.1) we get, also

$$\lambda = 0$$

$$\dot{\lambda} = 0$$

$$\ddot{\lambda} = 0$$

In this case (2.9) also reduce to –

(3.10) $\ddot{\upsilon} + \dot{\upsilon}^2 = -\dfrac{K^2}{8\pi} \exp(-4\upsilon)$ and

finally get,

(3.11) $e^{2\upsilon} = K_1 (t - t_0)^2 - \dfrac{K^2}{8\pi K_1}$

Other results same as equation (3.7) to (3.9).

In both cases – (I and II), finally the proper volume R^3, expansion scalar θ and shear σ are found to be

(3.12) $R^3 = K_1 (t - t_0)^2 - \dfrac{K^2}{8\pi K_1}$

(3.13) $\theta = 2K_1 (t - t_0)^2 R^{-3}$

(3.14) $\sigma^2 = \dfrac{1}{6} [2K_1(t - t_0)R^{-3}]^2$

From the above both solution we observe that at the initial epoch $(t - t_0)^2 = \dfrac{K^2}{8\pi K_1^2}$, the string model starts with an initial singularity, $R^3 \to 0$, while $\rho, \rho_p, \mathcal{T}, \theta, \sigma^2$ etc. diverge.

This is a line singularity, since $\exp(2\lambda) \to 1$ and $\exp(2\upsilon) \to 0$ (when m = 0). At a later instant,

When $(t - t_0)^2 = \dfrac{3K^2}{8\pi K_1}$

We have $\mathcal{T}_0 = 0$ & $\rho = \rho_p$. So at this epoch string vanish and we are left with a dust filled universe with a magnetic field. At this stage –

[3.15] $\rho = \dfrac{4\pi K_1^2}{K^2}$

[3.16] $R^3 = \dfrac{K^2}{4\pi K_1}$

[3.17] $\theta = \dfrac{2\sqrt{6\pi}}{K} = \dfrac{2(6\pi)^{1/2}}{K}$

[3.18] $\sigma^2 = \dfrac{2\pi K_1^2}{3K^2}$

i.e. All these parameters are of finite magnitude. In this solution matter is directly related with the magnetic field as in equation (3.9). When the magnetic field is absent the matter is also absent and the solution reduces to that of pure geometric string distribution.

Case III

Here we use $l = 0$ in equation [3.1]

$\lambda = 1 + m\,\upsilon$, then we get,

(3.19) $\lambda = m\,\upsilon$

By the use of (3.19) and (2.9) we get

(3.20) $(m + 1)\, \ddot{\upsilon} + (m^2 + m + 1)\, \dot{\upsilon}^2 = \dfrac{-K^2}{8\pi} \exp(-4\upsilon)$

This equation can be written as an integral equation

(3.21) $\displaystyle\int d\left[\dot{\upsilon}^2 \exp\left(2\left(\dfrac{m^2 + m + 1}{m + 1} \right)\upsilon \right) \right] = \dfrac{-K^2}{4\pi(m+1)} \int \exp\left(2\left(\dfrac{m^2 - m - 1}{m + 1} \right)\upsilon \right) d\upsilon + K_2$

Where K_2 is constant of integration. So we have

(3.22) $\dot{\upsilon}^2 = K_2 \exp\left[-2\left(\dfrac{m^2 - m - 1}{m + 1} \right)\upsilon \right] - \dfrac{K^2}{8\pi(m^2 - m - 1)} \exp(-4\upsilon)$

Which can again be written as an integral form as -

(3.23) $\displaystyle \int \frac{e^{2\upsilon}d\upsilon}{\left[K_2\,exp\left\{-2\left(\frac{m^2-m-1}{m+1}\right)\upsilon\right\}-\frac{K^2}{8\pi(m^2-m-1)}\right]^{1/2}}$

$= \pm (t - t_0)$

Where t_0 is another constant of integration. To solve (3.23) we choose m such that

(3.24) $\displaystyle \frac{m^2+2m+12}{m+3} = 5$

It is a quadratic equation in m which can be solved to give

(3.25) $\displaystyle m = \frac{3 \pm \sqrt{9+12}}{2}$ or

$\displaystyle m = \frac{3 \pm \sqrt{21}}{2}$

Using (3.25) in (3.23) followed by integration, we get

(3.26) $\displaystyle e^{2\upsilon} = \left[\frac{8\pi K_2}{K^2}\left(5 \pm \sqrt{21}\right) - \frac{K^2(t-t_0)^2}{2\pi(5\pm\sqrt{21})}\right]^{1/2}$

From (3.26) it is clear that arbitrary constant K_2 must be positive in this case. So we replace K_2 by δ^2 in the following. The other parameters can be found as before :-

(3.27) $\displaystyle \rho = \frac{\frac{9+2\sqrt{21}}{(5+\sqrt{21})^2}\cdot\frac{K^2}{16\pi}(t-t_0)^2 - (5+\sqrt{21})\delta^2}{\left[\frac{8\pi\delta^2}{K^2}(5+\sqrt{21}) - \frac{K^2(t-t_0)^2}{2\pi(5+\sqrt{21})}\right]^2}\cdot$

From energy condition, $\rho > 0$, which demands positive sign before $\sqrt{21}$ in equation (3.26) with this choice other parameters are found explicitly as follows :-

(3.28) $\displaystyle \mathrm{T} = \frac{9+\sqrt{21}\,\delta^2 - \frac{4+\sqrt{21}}{(5+\sqrt{21})^2}\cdot\frac{K^2}{16\pi^2}(t-t_0)^2}{\left[\frac{8\pi\delta^2}{K^2}(5+\sqrt{21}) - \frac{K^2(t-t_0)^2}{2\pi(5+\sqrt{21})}\right]^2}$

(3.29) $\displaystyle \rho_p = \frac{4\,\delta^2 + \frac{K^2(t-t_0)^2}{16\pi^2(5+\sqrt{21})}}{\left[\frac{8\pi\delta^2}{K^2}(5+\sqrt{21}) - \frac{K^2(t-t_0)^2}{2\pi(5+\sqrt{21})}\right]^2}$

(3.30) $\displaystyle \mathrm{R}^3 = \left[\frac{8\pi\delta^2(5+\sqrt{21})}{K^2} - \frac{K^2(t-t_0)^2}{2\pi(5+\sqrt{21})}\right] x$

$\displaystyle \frac{(7+\sqrt{21})}{8}$

(3.31) $\theta =$

$\displaystyle -\left(\frac{7+\sqrt{21}}{5+\sqrt{21}}\right)\left(\frac{K^2}{8\pi}\right)\left[\frac{t-t_0}{\frac{8\pi\delta^2}{K^2}(5+\sqrt{21}) - \frac{K^2(t-t_0)^2}{2\pi(5+\sqrt{21})}}\right]$

(3.32) $\displaystyle \sigma^2 = \frac{(7+\sqrt{21})}{(5+\sqrt{21})^2}\left(\frac{K^2}{48\pi^2}\right)$

$\displaystyle \left[\frac{t-t_0}{\frac{8\pi\delta^2}{K^2}(5+\sqrt{21}) - \frac{K^2(t-t_0)^2}{2\pi(5+\sqrt{21})}}\right]$

It is clear from (3.25) that

(3.33) $\displaystyle T^2 < \frac{16\pi^2\delta^2}{K^4}\left(5 + \sqrt{21}\right)^2$

Where T $= t - t_0$, when t $< t_0$ we have T > 0 and clearly.

From equation (3.31).

We have $\theta > 0$ and $\sigma^2 < 0$

Which indicates that our model is expanding because $\theta > 0$.

Case IV   When magnetic field is absent
(i,e   K = 0)

Then the field equation (2.7), (2.8) and (2.9) take the following forms

$$2\,\dot{\lambda}\dot{v} + \dot{v}^2 = \rho$$

$$2\,\ddot{v} + 3\,\dot{v}^2 = \top$$

$$\ddot{\lambda} + \dot{\lambda}^2 + \dot{v}^2 + \dot{\lambda}\dot{v} = 0$$

Now assuming (3.19) equation (2.9) reduces to

$$(m + 1)\,\ddot{v} + (m^2 + m + 1)\,\dot{v} = 0$$

Or,

[3.34]
$$\ddot{v} + \frac{m^2 + m + 1}{m + 1}\,\dot{v} = 0$$

This equation can be written as an integral equation -

[3.35]
$$\int d\left[\dot{v}\exp\left\{2\left(\frac{m^2+m+1}{m+1}\right)\right\} \times v\right] = \gamma$$

Where γ is constant of integration. So we have

[3.36]
$$\dot{v}^2 = \gamma\left[\exp\left\{-2\left(\frac{m^2+m+1}{m+1}\right)v\right\}\right]$$

Which can again be written as integral form as

[3.37]
$$\int \frac{e^{2v}\,.\,dv}{\left[\gamma\exp\left\{-2\left(\frac{m^2-m-1}{m+1}\right)v\right\}\right]^{1/2}} = \pm (t - t_0)$$

Where t_0 is the another constant of integration to solve (3.37) We get

(3.38)
$$e^{2v} = [\gamma(t - t_0)]^{\frac{2\,(m+1)}{m^2+m+1}}$$

(Where m is any constant) Therefore,

(3.39)
$$e^{2\lambda} = [\gamma(t - t_0)]^{\frac{2m\,(m+1)}{m^2+m+1}}$$

The physical parameters are found to be

(3.40)
$$R^3 = [\gamma(t - t_0)]^{\frac{(2+m)\,(m+1)}{m^2+m+1}}$$

(3.41)
$$\rho = \frac{2m+1}{(t - t_0)^2}$$

(3.42)
$$\top = \frac{1}{(t - t_0)^2}$$

(3.43)
$$\rho_p = \frac{2m}{(t - t_0)^2}$$

(3.44)
$$\theta = \frac{m+2}{(t - t_0)}$$

(3.45)
$$\sigma^2 = \frac{2}{3}\frac{(m-1)^2}{(t - t_0)^2}$$

Special Case

We choose m such that $m^2 + m - 2 = 0$ (quadratic equation) gives

(3.46)
$$m = 1 \text{ or } -2$$

We put the value of m using (3.46) in equation (3.38 – 3.45) and get : -

(3.47)
$$e^{2v} = [\gamma(t - t_0)]^{4/3} \quad \text{or}$$
$$[\gamma(t - t_0)]^{-2/3}$$

(3.48) $e^{2\lambda} = [\gamma(t - t_0)]^{4/3}$ or

$[\gamma(t - t_0)]^{4/3}$

The physical parameters are found to be

(3.49) $R^3 = [\gamma(t - t_0)]^2$ or

$[\gamma(t - t_0)]^0 = 1$

(3.50) $\rho = \dfrac{3}{(t - t_0)^2}$

or $\dfrac{-3}{(t - t_0)^2}$

(3.51) $T = \dfrac{1}{(t - t_0)^2}$

(3.52) $\rho_p = \dfrac{2}{(t - t_0)^2}$

or $\dfrac{-4}{(t - t_0)^2}$

(3.53) $\theta = \dfrac{3}{(t - t_0)}$

or   0

(3.54) $\sigma^2 = \dfrac{2}{3} \times 0 = 0$

or $\dfrac{2}{3} \times \dfrac{9}{(t - t_0)^2}$

Now if consider T = $t - t_0$ and $t > t_0$, then for the particular value of m like m $>$ -2, m $=$ -2 and m $<$ -2 we get different results for expansion.

If we consider m $>$ -2 in equation (3.44) then we get

(3.55) θ = +ve $\Rightarrow \theta > 0$,   which indicate our model is expanding

If m $= -2$ in equation (3.44) then

(3.56) $\theta = 0$, which indicate that neither expansion nor contraction occurs.

If m $<$ -2 in equation (3.44) we get

(3.57) $\theta = -$ve, $\Rightarrow$ $\theta < 0$, which indicate our model is contracted.

Again if consider T = $t - t_0$ and t < t$_o$, then for the particular value of m like m $>$ -2, m $= -2$ and m $<$ -2 we get different results for expansion.

If we consider m $>$ -2 in equation (3.44) then we get

(3.58) θ = -ve $\Rightarrow \theta$ < 0, which indicate our model is contracted.

If m $= -2$ in equation (3.44) then

(3.59) $\theta = 0$, which indicate that neither expansion nor contraction occurs.

If m $<$ -2 in equation (3.44) we get

(3.60) θ = +ve, $\Rightarrow$ $\theta > 0$, which indicate our model is expanding.

4. DISCUSSION : -

Here we show that at initial epoch t = t$_0$, $R^3 \rightarrow 0$, While $\rho, T, \rho_p \theta, \sigma^2$ all diverge and exp(2λ), exp (2υ) $\rightarrow 0$ the point of singularity. This is the starting point of the string model. Again at a later stage, when t $\rightarrow \infty$, $R \rightarrow \infty$ but all other physical parameters becomes insignificant. Further we see that for a pure geometric string ($\rho_p = 0$) model, we have to take m = 0. For this case the universe starts with string and ends up at a stage, where the massive strings themself disappear without any remnant. In the absence of magnetic field we have to take m > -2 (for t > t$_o$) then our modal is expanding. But if t<t$_o$ then the expansion occurs for m < -2. So the range of expansion of model (in absence of magnetic field) depends upon particular value of m and t respectively.

5. CONCLUSION : -

The present work extends the work of letelier and various other researcher. In addition we have consider a source with and without magnetic field respectively we have obtained four sets of solution with a special choice of metric coefficients, viz, $\lambda = 1 + m\upsilon$. For a particular value of m we observe that our model starts from a string dominated era but at a later instant string vanish and the universe become particle dominated. In this model gravitational field is coupled to magnetic field such that in the absence of the magnetic field (i.e K = 0) the system reduces to pure geometric string. For different value of m we calculate upper range of expansion of the string universe (For magnetic and without magnetic field), proper volume, shear scalar.

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