The Synthetic D Chart Under Non-Normality

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Abstract

The synthetic control chart based on Downton's estimator (Synthetic D chart) for monitoring the process variability developed by Rajmanya and Ghute (2013) is based on the assumption that the underlying process distribution is normal. In this paper, the performance of the synthetic D chart under non-normality is studied and is compared with the synthetic S chart. The comparison result shows that the synthetic D chart under non-normal distributions is more efficient than the synthetic S chart in detecting shifts in process variability.

Keywords- Downton's Estimator, Process variability, Non-normality, Average run length, Conforming run length, Control chart.

I. INTRODUCTION

Control charts are statistical process control tools that are widely used for controlling and monitoring a process. Shewhart's S chart is most widely used control chart to monitor process variability of a quality characteristic of interest. The S chart is based on the fundamental assumption that the underlying distribution of the quality characteristic is normal. It is easy to implement and is effective in the detection of large shifts in process standard deviation but become less effective for small shifts because it is based on only the most recent observation. Recently, Abbasi and Miller (2011) proposed the control chart based on Downton's estimate (D chart) of process standard deviation as an efficient alternative to S chart for monitoring process variability. It was shown that for normally distributed process, the D chart is equally efficient to the S chart for detecting shifts in the process variability.

In the literature synthetic charting concept is used by many researchers for improving the performance of the control charts. Huang and Chen (2005) developed synthetic S chart for process dispersion by combining the sample standard deviation, S and conforming run length (CRL) charts. The CRL chart is an attribute control chart proposed by Bourke (1991) for monitoring fraction nonconforming. With the objective of improving the performance of D chart, Rajmanya and Ghute (2013) developed synthetic D chart as a combination of D chart and CRL chart. It was shown that synthetic D chart is close competitor of synthetic S chart for detecting shifts in process standard deviation. The synthetic D chart was designed and evaluated under the assumption that the underlying process distribution is normal.

The behaviors of the commonly used control charts have been widely studied when some aspects of the assumption do not hold. One violation of the assumption is non-normality, which has been discussed by many authors. Borror et al. (1999) examined the performance of the EWMA chart for the mean in non-normal cases. Stoumbos and Reynolds (2000) studied the effect of non-normality and autocorrelation on the performance of various individual control charts for monitoring the process mean and/or variance. Calzada and Scariano (2001) have studied the robustness of the synthetic $\overline{X}$ chart when the normality assumption is violated and show that for sample size n = 6, the synthetic chart is robust to violations of the normality assumption. Kao and Ho (2007) examined the performance of R chart and found that R chart is robust to non-normality. Schoonhoven and Does (2010) studied the design schemes for the $\overline{X}$ control chart under non-normality.

The objective of this paper is to investigate the effect of non-normality on the statistical performance of the synthetic D control chart for monitoring process variability of a continuous process. Non-normal distribution can be a symmetric distribution or a skewed distribution. In this study, we selected a t distribution as symmetric distribution and Weibull distribution as the skewed distribution to examine and compare the performance of a synthetic D chart under non-normality with synthetic S chart.

II. MATERIALS AND METHODS

The D Control Chart

Let $X_1, X_2, ..., X_n$ represents a random sample of size n from a normally distributed process with process mean μ and standard deviation σ and $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ denote the corresponding order statistics. Downton (1966) proposed the following estimator of process standard deviation σ

$$D = \frac{2\sqrt{\pi}}{n(n-1)} \sum_{i=1}^{n} \left(i - \frac{1}{2}(n+1) \right) X_{(i)} \qquad (1)$$

which is unbiased estimator of σ for normally distributed quality characteristics. Let μ_0 and σ_0 be the in-control values of μ and σ respectively. When a shift in process standard deviation occurs, we have change from the in-control value σ_0 to the out-of-control value $\sigma_1 = \delta \sigma_0$ $(0 < \delta \ne 1)$. When $\delta = 1$, the process is considered to be in-control . For $\delta > 1$ an increase in σ occurs, process is considered to be out-of-control and an upper control limit $k^+\sigma_0$ of D chart is required, and a signal is issued if $D > k^+\sigma_0$. For $\delta < 1$ decrease in σ occurs, process is considered to be out-of-control and lower limit $k^-\sigma_0$ of D chart is required, and a signal is issued if $D < k^-\sigma_0$. In this study we consider the case of increase in the process standard deviation.

The average run length (ARL) which denotes the average number of D samples required to detect a change in σ of the D chart can be calculated as

$$\begin{aligned} ARL_D(\delta) &= \frac{1}{\Pr(D > k^+\sigma_0 \mid \sigma = \delta\,\sigma_0)} \\ &= \frac{1}{\Pr\left(Z > \frac{k^+}{\delta} \right)} \\ &= \frac{1}{1 - F\left(\frac{k^+}{\delta} \right)} \qquad (2) \end{aligned}$$

where, $Z = \frac{D}{\sigma}$ and $F(.)$ its cumulative distribution function.

Synthetic D Chart

The synthetic D control chart developed by Rajmanya and Ghute (2013) is a combination of the D chart and the CRL chart. The synthetic D chart consists of D sub-chart and CRL sub-chart. For detecting increase in standard deviation, the D sub-chart has upper control limit $UCL = k^{+}\sigma_0$. The CRL sub-chart has a lower control limit L. These limits are called as design parameters of the synthetic D chart. According to the synthetic procedure, the signal is based on the CRL. The CRL is the number of conforming samples between two consecutive nonconforming samples including the end nonconforming sample.

Let $ARL_S(\delta)$ denote the average number of D samples required for a synthetic D chart to signal a shift of magnitude δ in process standard deviation σ. Then the ARL values of the synthetic D chart for a given shift of magnitude δ is given by the formula,

$$ARL_S(\delta) = \frac{1}{1 - F\left(\frac{k^{+}}{\delta}\right)} \times \frac{1}{\left[1 - F\left(\frac{k^{+}}{\delta}\right)\right]^{L-1}} \quad (3)$$

When $\delta = 1$, in-control ARL of the synthetic D chart is

$$ARL_S(1) = \frac{1}{1 - F(k^{+})} \times \frac{1}{\left[1 - F(k^{+})\right]^{L-1}} \quad (4)$$

To design synthetic |D| control chart means to obtain two design parameters L and k^{+} suitably that guarantee a minimum ARL for predetermined shift of magnitude δ^*. The design shift δ^* is the magnitude considered larger enough to seriously impair the quality of the products; thus corresponding $ARL_S(\delta^*)$ should be as small as possible. $ARL_S(1)$ is decided by the requirement on the false alarm rate. The synthetic D control chart is properly designed by solving an optimization problem. The objective function

$$ARL_S(\delta^*) = \text{minimum}, \quad (5)$$

subject to the equality constraint in Eq. (4).

The optimal design procedure for the synthetic D chart uses the approach described in the following steps.

(1) Specify *n*, δ^* and in-control ARL.

(2) Initialize L as 2.

(3) Obtain k^+ by solving Eq.(4).

(4) Calculate ARL for δ^* from current value of L and k^+ using Eq.(3) (use $\delta = \delta^*$).

(5) If L is not equal to 2, go to the next step; otherwise, L is increased by one and go back to step (3).

(6) If the current ARL for δ^* is greater than the preceding one, go to the next step; otherwise, L is increased by one and go back to step (3).

(7) Take the preceding L and k^+ as the optimal design parameters for the synthetic chart.

To illustrate the design of the synthetic D chart, an example with n =10, $\delta^* = 1.4$ and in-control $ARL_S(1) = 200$. Table 1 shows that each set of (L, k^+) results in different $ARL_S(\delta^*)$. The ARL first declines and then increases. The $ARL_S(\delta^*)$ reaches its minimum at 3.0926 when L = 7 and k = 1.5180. So in this case the design parameters of the synthetic D chart are L = 7 and k = 1.5180.

Table-1. Sets of (L, k^+) and corresponding ARL for $\delta^* = 1.4$.

(L, k^+)	ARL for δ^*
(2, 1.4098)	4.5127
(3, 1.4530)	3.5726
(4, 1.4780)	3.2783
(5, 1.495)	3.1542
(6, 1.5073)	3.1005
(7, 1.5180)	**3.0926**
(8, 1.5264)	3.0999
(9, 1.5340)	3.1252
(10, 1.5400)	3.1473

The optimal values of L and k for sample size n = 5, 8 and 10 provided in-control ARL = 200 and 370 under $\delta^* =$ 1.2, 1.4 and 1.6 are presented in Table 2.

Table-2. Optimal design parameters of the synthetic D chart.

n	δ^*	$ARL_S(1) = 200$		$ARL_S(1) = 370$	
		L	k	L	k
5	1.2	17	1.843	24	1.927
	1.4	10	1.786	11	1.856
	1.6	7	1.750	8	1.825
8	1.2	12	1.595	18	1.658
	1.4	7	1.550	9	1.612
	1.6	5	1.522	5	1.567
10	1.2	12	1.5192	15	1.5638
	1.4	6	1.469	7	1.518
	1.6	4	1.438	5	1.495

Effect of Non-normality on Synthetic D chart

The synthetic D control chart developed for monitoring the process variability is based on the assumption that the underlying process distribution is normal. In this section, we examine the effects of non-normality on the statistical performance of the synthetic D control chart. In order to study the effect of non-normality, we considered heavy-tailed symmetric and skewed distributions. Specifically, we simulated observations in the heavy tailed symmetric case from t distribution and in the skewed case from the Weibull distribution.

The probability density function of t distribution with ν degrees of freedom is

$$f(x) = \frac{\Gamma\left[(\nu+1)/2\right]}{\sqrt{\nu\pi}\;\;\Gamma(\nu/2)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad , -\infty < x < \infty\,;\; \nu > 0. \qquad (6)$$

The probability density function of Weibull distribution with shape parameter α and scale parameter β is

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \quad , \; x \ge 0\,;\; \alpha > 0,\; \beta > 0 \qquad (7)$$

III. RESULTS AND DISCUSSIONS

Performance comparisons for Non-normal process

The synthetic D chart is designed by using the values of design parameters (L, k) with $\delta^* = 1.2$ from Table 2, which are, technically, only appropriate for normally distributed process data. The design parameter k of the charts is then adjusted so that both synthetic charts have approximately the same in-control ARL value 200. The ARL values of the synthetic S and synthetic D charts are computed using 10000 simulations when underlying process distribution is normal, heavy tailed and skewed. To study the effect of non-normality on the performance of the synthetic D control chart, we have considered the process data from t distribution with $\nu = 5$ and Weibull distribution with shape parameter $\alpha = 2$ and scale parameter $\beta = 1$.

Table 3 presents the ARL profiles of the synthetic S and synthetic D charts for increase in the process standard deviation when underlying process data follows standard normal distribution with in-control ARL = 200, $\delta^* = 1.2$ and sample sizes n = 5, 8 and 10.

Table-3. ARL comparison of synthetic charts under normal distribution.

Shift δ	n = 5		n = 8		n = 10	
	Syn S k =1.722 L = 18	Syn D k =1.843 L = 17	Syn S k =1.5263 L = 12	Syn D k =1.595 L = 12	Syn S k =1.4663 L = 12	Syn D k =1.5192 L = 12
1.0	200	200	199	200	200	201
1.1	42.62	43.92	33.29	32.80	27.96	28.61
1.2	15.54	15.89	10.49	10.75	8.58	8.66
1.3	8.09	8.34	5.07	5.27	4.22	4.41
1.4	5.25	5.38	3.31	2.49	2.77	2.89
1.5	3.86	3.92	2.50	2.56	2.09	2.16
2.0	1.77	1.79	1.28	1.31	1.18	1.18

From Table 3, when underlying process distribution is normal, we observe that for any range of shifts, the synthetic S control chart produces slightly smaller out-of-control ARL than that of the synthetic D chart. The synthetic S chart performs slightly better than the synthetic D chart.

Table 4 and Table 5 present the ARL profiles of the synthetic S and synthetic D control charts for increase in the process standard deviation when underlying process data follows t distribution and Weibull distribution respectively each with in-control ARL = 200, $\delta^* = 1.2$ and sample sizes n = 5, 8 and 10.

Table-4. ARL comparison of synthetic charts under t distribution.

Shift δ	n = 5		n = 8		n = 10	
	Syn S k =2.0535 L = 18	Syn D k =2.10 L = 17	Syn S k =1.78138 L = 12	Syn D k =1.771 L = 12	Syn S k =1.746 L = 12	Syn D k =1.6706 L = 12
1.0	200	200	200	200	200	200
1.1	83.25	74.89	73.64	63.31	70.28	56.08
1.2	39.25	34.84	31.79	24.73	28.47	21.18
1.3	21.71	18.91	16.11	12.64	14.09	10.16
1.4	13.51	11.88	9.42	7.50	8.04	5.99
1.5	9.29	7.96	6.30	4.97	5.32	4.08
2.0	3.18	2.93	2.12	1.91	1.85	1.63

Table-5. ARL comparison of synthetic charts under Weibull distribution.

Shift δ	n = 5		n = 8		n = 10	
	Syn S k =1.7678 L = 18	Syn D k =1.8475 L = 17	Syn S k =1.5648 L = 12	Syn D k =1.5935 L = 12	Syn S k =1.5014 L = 12	Syn D k =1.5152 L = 12
1.0	200	200	200	200	200	200
1.1	47.51	44.32	36.61	33.76	32.03	29.07
1.2	18.05	16.44	12.19	11.23	9.97	9.14
1.3	9.28	8.63	5.91	5.68	5.00	4.54
1.4	5.98	5.61	3.84	3.52	3.14	3.01
1.5	4.35	4.07	2.76	2.65	2.34	2.25
2.0	1.88	1.85	1.36	1.33	1.22	1.19

From Table 4 and Table 5, when underlying process distribution is heavy tailed and skewed we observe that, the synthetic D chart consistently produces smaller out-of-control ARL values than that of the synthetic S chart for entire range of shifts in the process standard deviation.

Conclusions

In this paper, a synthetic D chart based on Downton's estimator is studied under non-normality of process data. By comparing synthetic S and synthetic D charts when process distribution is heavy tailed or skewed, we found that the synthetic D chart detects the shifts in process standard deviation quicker than the synthetic S chart. Hence for non-normal processes the synthetic D chart is uniformly better than the synthetic S chart.

References

1. Abbasi, S. A. and Miller, A. (2011). D chart: An efficient alternative to monitor process dispersion, *Proceedings of the World Congress on Engineering and Computer Science* (WCECS 2011), Vol II, San Francisco, USA.

2. Borror, C.M., Montgomery, D.C. and Runger, G.C. (1999). Robustness of the EWMA. control chart to non-normality. *Journal of Quality Technology*, 31, 309-316.

3. Bourke, P. D. (1991). Detecting a shift in fraction nonconforming using run-length control charts with 100% inspection. *Journal of Quality Technology*, 23(3), 225- 238.

4. Calzada, M. E. and Scariano, S. M. (2001). The robustness of the synthetic control chart to non-normality. *Communications in Statistics- Simulation and Computation*, 30(2), 311-326.

5. Downton, F. (1966), Linear estimates with polynomial coefficients, *Biometrika*, 53, 129-141.

6. Huang, H. J. and Chen, F. L. (2005). A synthetic control chart for monitoring process dispersion with sample standard deviation. *Computers and Industrial Engineering*, 49(2), 221-240.

7. Kao, S. C. and Ho, C. (2007). Robustness of R chart to non-normality, *Communications in Statistics-Simulation and Computation*, 38, 541-557.

8. Rajmanya, S. V. and Ghute, V. B. (2013). A synthetic control chart for monitoring process variability. *Quality and Reliability Engineering International* (Revised and submitted)

9. Schoonhoven, M. and Does, R. J. (2010). The $\overline{X}$ chart under non-normality, *Quality and Reliability Engineering International*, 26, 167-176.

10. Stoumbos, Z. G. and Reynolds, M. R. Jr. (2000). Robustness to non-normality and auto-correlation on the individual control charts, *Journal of Statistical Computation and simulation*, 66, 145-187.