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Theorem for Expansion Mapping Without Continuity in Cone Metric Space

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Abstract

In the present paper we prove a unique common fixed point theorem for expansion mapping without continuity for four self mappings. We use the condition of weakly compatibility to prove the fixed point. In this result the cone is not necessarily a normal cone. The result is an extension and generalisations of many results available in the literature

Keywords: *Cone metric space, Expansion mapping, weakly compatible mappings.*

1.Introduction

In 1922, Banach proved a common fixed point theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. This result of Banach is known as Banach's fixed point theorem or Banach contraction principle. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways.

In 1984, Wang et al. [10] presented some interesting work on expansion mappings in metric spaces which correspond to some contractive mappings in [7].Further, Khan et al. [4] generalized the result of [10] by using functions. Also, Rhoades [8] and Taniguchi [9] generalized the results of Wang [10] for a pair of mappings. Kang [3] generalized the result of Khan et al. [4], Rhoades [8] and Taniguchi [9] for expansion mappings. Daffer and Kaneko [1] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces.

Recently, Huang and Zhang [2] introduced the concept of a cone metric space as a generalization of a metric space. They proved the properties of sequences in cone metric spaces and obtained various fixed point theorems for contractive mappings.

We have proved fixed point theorem for expansion mapping for four mapping in cone metric space. The theorem is an extensions and generalizations of Yan Han and Shaoyuan Xu [13], Wasfi Shatanawi and Fadi Awawdeh [11], Xianjiu Huang, Chuanxi Zhu and Xi Wen[12]

2. Preliminary

We need to use the following fundamental concepts throughout this paper.

2.1 Cone

Let E be a real Banach space and $P \subset E$ .Then the set P is called a cone if and only if
(i) P is closed, non empty and $P \neq \phi$;
(ii) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$
(iii) $P \cap (-P) = \phi$.

2.2 Partial ordered cone

For given cone $P \subset E$,we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ for $y - x \in P_0$, where P_0 stands for interior of P. Also we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$.

2.3 Cone metric space

Let X be a non empty set. Suppose that the mappings $d : X \times X \rightarrow E$ satisfies:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and

$d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ .Then d is called a cone metric on X and (X ,d) is called a cone metric space.

2.4 Example 1

Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \rightarrow E$ such that

$d(x, y) = (x - y, \beta(x - y))$, here $\beta \geq 0$ is a constant.

Then (X, d) is a cone metric space

2.5 Expansion mapping

Let (X, d) be a complete cone metric space. If f is a mapping of X into itself and if there exists a constant q > 1 such that $d(f(x), f(y)) \geq q\, d(x, y)$ for each $x, y \in X$, then f is called as the expansion mapping in X.

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2.6 Convergent Sequence

Let (X, d) be a cone metric space. The sequence $\{x_n\}$ in X is said to be a convergent sequence if for every c ϵ E with 0 < c, there is n_0 ϵ N such that for all $n \geq n_0$, $d(x_n, x) < c$ for some x ϵ X. We denote this by $\lim_{n \to \infty} x_n = x$.

2.7 Cauchy Sequence

Let (X, d) be a cone metric space. The sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for all c ϵ E with $0 \ll c$, there is n_o ϵ N such that $d(x_m, x_n) \ll c$, for all $m, n \geq n_0$.

2.8 Complete cone metric space

A cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

2.9 Weakly Compatible

Let f and g be two self-maps defined on a set X . Then f and g are said to be weakly compatible if they commute at coincidence points. That is, if fu =gu for some $u \in X$, then fgu=gfu.

2.10 Coincidence Point

Let f and g be self-maps on a set X . If w = fx = gx, for some x in X, then w is called coincidence point of f and g.

Our theorem is an extension and generalization of Yan Han and Shaoyuan Xu [13], Wasfi Shatanawi and Fadi Awawdeh [11], Xianjiu Huang, Chuanxi Zhu and Xi Wen[12]

3. Main theorem

Let (X, d) be a complete cone metric space. Suppose A, B, P, Q are self mappings on X itself and each of it are surjective.

(i) $A(X) \subseteq Q(X)$, $B(X) \subseteq P(X)$.

(ii) (A,P) and (B,Q) are weakly compatible.

(iii) Suppose for $\alpha, \beta, \gamma, \delta, \theta$ such that $\alpha, \beta, \gamma, \delta, \theta \in [0,1)$ and $\alpha + \beta + \gamma > 1$

$$d(Px, Qy) \geq \alpha\, d(Ax, Px) + \beta\, d(By, Qy) + \gamma\, d(Ax, By) + \delta\, d(Ax, Qy) + \theta\, d(Px, By) \quad ......(1)$$

for all $x.y \in X$. Either $1 + \theta > \alpha$ or $1 + \delta > \beta$ Then A, B, P, Q has a unique common fixed point in X.

Proof: Let x_0 is an arbitrary point in X.

∵ A, B, P, Q are surjective.

∴ There exists $\{x_{2n}\}$,$\{y_{2n}\} \in X$ such that $A x_{2n} = Q x_{2n+1} = y_{2n}$ and $B x_{2n+1} = P x_{2n+2} = y_{2n+1}$ for all n.

Case-1:

Putting $x = x_{2n}$, $y = x_{2n+1}$ in (1) , we get

$$d(Px_{2n}, Qx_{2n+1}) \geq \alpha\, d(Ax_{2n}, Px_{2n}) + \beta\, d(Bx_{2n+1}, Qx_{2n+1}) + \gamma\, d(Ax_{2n}, Bx_{2n+1}) + (Ax_{2n}, Qx_{2n+1}) + \theta\, d(Px_{2n}, Bx_{2n+1}) ....(2)$$

$$\therefore d(y_{2n-1}, y_{2n}) \geq \alpha\, d(y_{2n}, y_{2n-1}) + \beta\, d(y_{2n+1}, y_{2n}) + \gamma\, d(y_{2n}, y_{2n+1}) + \delta\, d(y_{2n}, y_{2n}) + \theta\, d(y_{2n-1}, y_{2n+1})$$

$$\therefore d(y_{2n-1}, y_{2n}) \geq \alpha\, d(y_{2n}, y_{2n-1}) + \beta\, d(y_{2n+1}, y_{2n}) + \gamma\, d(y_{2n}, y_{2n+1}) + \theta\left(d(y_{2n+1}, y_{2n}) - d(y_{2n}, y_{2n-1})\right)$$

$$(1-\alpha+\theta)\; d(y_{2n}, y_{2n-1}) \geq (\beta+\theta+\gamma)\; d(y_{2n+1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{(1-\alpha+\theta)}{(\beta+\theta+\gamma)}\; d(y_{2n}, y_{2n-1})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{(1-\alpha+\theta)}{(\beta+\theta+\gamma)}\; d(y_{2n}, y_{2n-1})$$

here $(1-\alpha+\theta) > 0 \Rightarrow 1+\theta > \alpha$ .....(3)

and $(\beta+\theta+\gamma) > (1-\alpha+\theta) \Rightarrow \alpha+\beta+\gamma > 1$

$$d(y_{2n}, y_{2n+1}) \leq h\; d(y_{2n}, y_{2n-1})$$

where $h = \frac{(1-\alpha+\theta)}{(\beta+\theta+\gamma)}$ and $h < 1$ ....(4)

Case-II:

Putting $x = x_{2n}, y = x_{2n-1}$ in (1) , we get

$$d(Px_{2n}, Qx_{2n-1}) \geq \alpha\, d(Ax_{2n}, Px_{2n}) + \beta\, d(Bx_{2n-1}, Qx_{2n-1}) + \gamma\, d(Ax_{2n}, Bx_{2n-1}) + \delta\, d(Ax_{2n}, Qx_{2n-1}) + \theta\, d(Px_{2n}, Bx_{2n-1})$$

$$d(y_{2n-1}, y_{2n-2}) = d(Px, Qx_{2n-1}) \geq \alpha\, d(y_{2n}, y_{2n-1}) + \beta\, d(y_{2n-1}, y_{2n-2}) + \gamma\, d(y_{2n}, y_{2n-1}) + \delta\, d(y_{2n}, y_{2n-2}) + \theta\, d(y_{2n-1}, y_{2n-1}) \quad .....(5)$$

$$\therefore d(y_{2n}, y_{2n-1}) \geq \alpha\, d(y_{2n}, y_{2n-1}) + \beta\, d(y_{2n-1}, y_{2n-2}) + \gamma\, d(y_{2n}, y_{2n-1}) + \delta\, d(y_{2n}, y_{2n-2}) + \theta\, d(y_{2n-1}, y_{2n-1})$$

$$\because d(y_{2n}, y_{2n-2}) \geq d(y_{2n}, y_{2n-1}) - d(y_{2n-1}, y_{2n-2})$$

$$\therefore d(y_{2n}, y_{2n-1}) \geq \alpha\, d(y_{2n}, y_{2n-1}) + \beta\, d(y_{2n-1}, y_{2n-2}) + \gamma\, d(y_{2n}, y_{2n-1}) + \delta\left(d(y_{2n}, y_{2n-1}) - d(y_{2n-1}, y_{2n-2})\right)$$

$$(1-\beta+\delta)\; d(y_{2n-1}, y_{2n-2}) \geq (\alpha+\gamma+\delta)\; d(y_{2n}, y_{2n-1})$$

$$d(y_{2n}, y_{2n-1}) \leq \frac{(1-\beta+\delta)}{(\alpha+\gamma+\delta)}\; d(y_{2n-1}, y_{2n-2})$$

here $(1-\beta+\delta) > 0 \Rightarrow (1+\delta) > \beta$ ..(6)

$(\alpha+\gamma+\delta) > (1-\beta+\delta) \Rightarrow \alpha+\beta+\gamma > 1$

$$d(y_{2n}, y_{2n-1}) \leq k\; d(y_{2n-1}, y_{2n-2})$$

where $k = \frac{(1-\beta+\delta)}{(\alpha+\gamma+\delta)}$ and $k < 1$. ....(7)

$\therefore$ From case - I or case - II, we get

$$d(y_{2n+1}, y_{2n}) \leq h\, d(y_{2n}, y_{2n-1}) \leq h\,k\, d(y_{2n-1}, y_{2n-2})$$

.

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$$d(y_{2n+1}, y_{2n}) \leq (kh)^n\, d(y_1, y_0)$$

and

$$d(y_{2n}, y_{2n-1}) \leq k\, d(y_{2n-1}, y_{2n-2}) \leq k\,h\, d(y_{2n-2}, y_{2n-3}) \leq k(hk)\, d(y_{2n-3}, y_{2n-4})$$

.

.

$$d(y_{2n}, y_{2n-1}) \leq k(kh)^n\, d(y_1, y_0)$$

$\because k < 1$ and $h < 1 \Rightarrow kh < 1$ i.e. $kh \in [0,1)$

Hence for $n > m$

$$d(y_{2n+1}, y_{2m-1}) \leq d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1}) + \dots + d(y_{2m}, y_{2m-1})$$

$$\leq \sum_{i=n}^{m} (kh)^i\; d(y_1, y_0) + \sum_{i=n}^{i=m} k\; (kh)^i\; d(y_1, y_0)$$

$$\leq \frac{(kh)^m \left(1-(kh)^{n-m+1}\right)}{1-kh} d(y_1, y_0) + k\frac{(kh)^m \left(1-(kh)^{n-m+1}\right) d(y_1,y_0)}{1-kh}$$

$$= \frac{(1+k)(kh)^m \left(1-(kh)^{n-m+1}\right) d(y_1,y_0)}{1-kh} \to 0$$

as $n, m \to \infty$

For $c \gg 0$, we can find some $\in > 0$ such that $c - x \in \operatorname{int} P$, where $\|x\| < \in$ i.e. $x \ll c$.

For this $\in$, we can find a natural number N such that

$$\left|\frac{(1+k)(kh)^m \left(1-(kh)^{n-m+1}\right) d(y_1,y_0)}{1-kh}\right| < \in \text{ for } n, m > N.$$

Thus we get $d(y_{2n+1}, y_{2m-1}) \ll c$ for $n > m > N$

Thus $\{y_n\}$ is a cauchy sequence.

As X is complete, there exists some $y \in X$ such that $y_n \to y \in X$. It is equivalent to say that $y_{2n} \to y \in X$ and $y_{2n+1} \to y \in X$.

$A x_{2n} = Q x_{2n+1} = y_{2n} \to y \in X$. and

$B x_{2n+1} = P x_{2n+2} = y_{2n+1} \to y \in X$.

A,B,P,Q are onto mappings thus there exists p, q, r, $s \in X$ such that $Ap = Qq = y$ and $Br = Ps = y$

Now we will show that $p = q = r = s = y$

Putting $x = x_{2n}$ and $y = q$ in (1) we get,

$$d(Px_{2n}, Qq) \geq \alpha\, d(Ax_{2n}, Px_{2n}) + \beta\, d(Bq, Qq) + \gamma\, d(Ax_{2n}, Bq) + \delta\, d(Ax_{2n}, Qq) + \theta\, d(Px_{2n}, Bq)$$

As $n \to \infty$

$$d(y, y) \geq \alpha\, d(y, y) + \beta\, d(Bq, y) + \gamma\, d(y, Bq) + \delta\, d(y, y) + \theta\, d(y, Bq)$$

$$\therefore 0 \geq (\beta + \gamma + \theta)\, d(y, Bq)$$

$\therefore$ $d(y,Bq) = 0$

$\therefore$ $Bq = y$

$\because$ $Bq = y \Rightarrow Bq = Qq$

As (B, Q) are weakly compatible

$\therefore$ $BQ q = Q Bq$

$\therefore$ $By = Qy$

Again putting $x = x_{2n}$ in (1) we get

$$d(Px_{2n}, Qy) \geq \alpha\, d(Ax_{2n}, Px_{2n}) + \beta\, d(By, Qy) + \gamma\, d(Ax_{2n}, By) + \delta\, d(Ax_{2n}, Qy) + \theta\, d(Px_{2n}, By)$$

as $n \to \infty$ we get

$$d(y, By) \geq \alpha\, d(y, y) + \beta\, d(By, By) + \gamma(y, By) + \delta\, d(y, By) + \theta\, d(y, By)$$

$$d(y, By) \geq \alpha\, 0 + \beta\, 0 + \gamma(y, By) + \delta\, d(y, By) + \theta\, d(y, By)$$

$\therefore$ $(1-(\gamma+\delta+\theta))\, d(y,By)=0$ $\quad\therefore$ $By = y$

$\therefore$ $By = Qy = y$

Now, putting x=p in (1) we get

$$d(Ps, Qy) \geq \alpha\, d(As, Ps) + \beta\, d(By, Qy) + \gamma\, d(As, By) + \delta\, d(As, Qy) + \theta\, d(Ps, By) \quad \therefore$$

$$d(y, y) \geq \alpha\, d(Ap, y) + \beta\, d(y, y) + \gamma\, d(Ap, y) + \delta\, d(Ap, y) + \theta\, d(y, y) \quad \therefore$$

$$(\alpha + \gamma + \delta)\, d(Ap,y) \leq 0$$

$\therefore$ $Ap = y$

$\because$ (A, P) are weakly compatible

$\therefore$ $APp=PAp \Rightarrow Ay = Py$

Now put putting x = y in (1) we get

$$d(Py, Qy) \geq \alpha\, d(Ay, Py) + \beta\, d(By, Qy) + \gamma\, d(Ay, By) + \delta\, d(Ay, Qy) + \theta\, d(Py, By)$$

$$d(Ay, y) \geq \alpha\, d(Ay, Ay) + \beta\, d(y, y) + \gamma\, d(Ay, y) + \delta\, d(Ay, y) + \theta\, d(Ay, y)$$

$\therefore$ $d(Ay,y)=0$

$\therefore$ $Ay=y$

$\therefore$ $Ay=By=Py=Qy=y$

Now to prove the uniqueness of the fixed point Let if possible there are two fixed points say y and y*

Ay = By = Py = Qy =y and A y* = B y*= P y* = Q y* = y*

Putting x = y and y = y* in (1),we get

$$d(Py, Qy^*) \geq \alpha\, d(Ay, Py) + \beta\, d(By^*, Q y^*) + \gamma\, d(Ay, B y^*) + \delta\, d(Ay, Q y^*) + \theta\, d(Py, B y^*)$$

$$d(y, y^*) \geq \alpha\, d(y, y) + \beta\, d(y^*, y^*) + \gamma\, d(y, y^*) + \delta\, d(y, y^*) + \theta\, d(y, y^*)$$

$$\therefore 0 \geq ((\gamma + \delta + \theta) - 1) d(y, y^*)$$

which is a contradiction.Thus $y = y^*$

i.e. fixed point is unique.

3.1 Corollary-1.

Let (X, d) be a complete cone metric space. Suppose A, P are self mappings on X itself and each of it are surjective.
(i)A(X) $\subseteq$ P(X),
(ii)(A,P) are weakly compatible.
(iii)Suppose for $\alpha, \beta, \gamma, \delta, \theta$ such that $\alpha, \beta, \gamma, \delta, \theta \in [0,1)$ and $\alpha + \beta + \gamma > 1$

$$d(Px, Py) \geq \alpha\, d(Ax, Px) + \beta\, d(Ay, Py) + \gamma\, d(Ax, Ay) + \delta\, d(Ax, Py) + \theta\, d(Px, Ay)$$

for all x.y $\in$ X. Either $1 + \theta > \alpha$ or $1 + \delta > \beta$ Then A, P have a unique common fixed point in X.

Proof: In Theorem -1, if we put P=Q and A=B, we get the proof.

3.2 Corollary-2.

Let (X, d) be a complete cone metric space. Suppose P is self mapping on X itself and it are surjective. Suppose for $\alpha, \beta, \gamma, \delta, \theta$ such that $\alpha, \beta, \gamma, \delta, \theta \in [0,1)$ and $\alpha + \beta + \gamma > 1$

$$d(Px, Py) \geq \alpha\, d(x, Px) + \beta\, d(y, Py) + \gamma\, d(x, y) + \delta\, d(x, Py) + \theta\, d(Px, y)$$

for all x. y $\in$ X. . Either $1 + \theta > \alpha$ or $1 + \delta > \beta$ Then A, P have a unique common fixed point in X.

Proof: Putting A=I, identity mapping we get the proof.

3..3 Corollary-3.

Let (X, d) be a complete cone metric space. Suppose A, P are self mappings on X itself and each of it are surjective.
(i)A(X) $\subseteq$ P(X)
(ii)(A, P) are weakly compatible.
(iii)Suppose for α, β, γ such that $\alpha, \beta, \gamma \in [0,1)$ and $\alpha + \beta + \gamma > 1$
$d(Px, Py) \geq \alpha\, d(Ax, Px) + \beta\, d(Ay, Py) + \gamma\, d(Ax, Ay)$ for all x. y$\in$ X .Then A, P have a unique common fixed point in X.

Proof: In the corollary-1 if we put $\delta = \theta = 0$ we get the proof.

3.4 Corollary-4.

Let (X, d) be a complete cone metric space. Suppose P is self mapping on X itself and it is surjective. Suppose for α, β, γ such that $\alpha, \beta, \gamma \in [0,1)$ and $\alpha + \beta + \gamma > 1$
$d(Px, Py) \geq \alpha(Px, x) + \beta\, d(Py, y) + \gamma\, d(x, y)$ for all x.y$\in$ X . Then P has a unique common fixed point in X.

Proof: In the corollary-3, if we put A=I, identity mapping, we get the proof.

3.5 Corollary-5.

Let (X, d) be a complete cone metric space. Suppose P is self mapping on X itself and it is surjective.
For $\gamma > 1$.
$d(Px, Py) \geq \gamma\, d(x, y)$ for all x.y$\in$ X . Then P has a unique common fixed point in X.

Proof: In the corollary-4, if we put $\alpha = \beta = 0$, we get the proof.

3.6 Corollary-6.

Let (X, d) be a complete cone metric space. Suppose A, P are self mappings on X itself and each of it are surjective.
(i)A(X) $\subseteq$ P(X)
(ii)(A, P) are weakly compatible.

(iii)Suppose for such that $\gamma > 1$
$d(Px, Py) \geq \gamma\, d(Ax, Ay)$ for all x. $y \in X$ .Then A, P have a unique common fixed point in X.

Proof: In the corollary-1, if we put $\alpha = \beta = \delta = \theta$, we get the proof.

4. Remark

(1) Corrollary-2 is the main result of Yan Han and Shaoyuan Xu [13]
(2) Corrollary-3 is the main theorem of Wasfi Shatanawi and Fadi Awawdeh [11]
(3) Corrollary-4 is the theorem 2.1 from Xianjiu Huang, Chuanxi Zhu and Xi Wen[12]
(4) Corrollary-6 is theorem 2.3 from Xianjiu Huang, Chuanxi Zhu and Xi Wen[12].

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6. References

[1] Daffer Z.P., Kaneko H., On Expansive Mappings, *Math. Japonica*, 7(1992), 733–735.
[2] Huang, L-G, Zhang, X: Cone metric space and fixed point theorems of contractive mappings. *J. Math. Anal. Appl*. (2007) 332,1468-1476
[3] Kang, SM: Fixed Points for Expansion Mappings. *Math. Jpn.* (1993) 38, 713-717
[4] Khan, MA, Khan, MS, Sessa, S: Some Theorems On Expansion Mappings And Their Fixed Points. *Demonstr. Math.* (1986)19,673-683
[5] Kumar S., Common Fixed Point Theorems For Expansion Mappings In Various Spaces, *Acta Math. Hungar.*, 118 (2008), 9–28.
[6] Kumar S.,Garg S.K., Expansion Mapping Theorems In Metric Spaces, *Int.J. Contemp. Math. Sciences*, 4 (2009), 1749–1758.
[7] Rhoades, BE: A Comparison Of Various Definitions Of Contractive Mappings. *Trans. Am. Math. Soc.* 226(1977), 257-290
[8] Rhoades, BE: Some Fixed Point Theorems For Pairs Of Mappings. *Jnanabha* 15(1985), 151-156
[9] Taniguchi, T: Common Fixed Point Theorems On Expansion Type Mappings On Complete Metric Spaces. *Math. Jpn.* 34(1989),139-142
[10] Wang, SZ, Li, BY, Gao, ZM, Iseki, K: Some Fixed Point Theorems On Expansion Mappings. *Math. Jpn.* 29(1984), 631-636
[11] Wasfi Shatanawi and Fadi Awawdeh, Some Fixed And Coincidence Point Theorems ForExpansive Maps In Cone Metric Spaces, *Fixed Point Theory and Applications* 2012,2012:19 http://www.fixedpointtheoryandapplications.com/content/2012/1/19
[12] Xianjiu Huang, Chuanxi Zhu and Xi Wen, Fixed Point Theorems For ExpandingMappings In Cone Metric Spaces, *math. reports* 14(64), 2 (2012), 141–148
[13] Yan Han and Shaoyuan Xu, Some New Theorems Of Expanding Mappings Without Continuity In Cone Metric Spaces *Fixed Point Theory and Applications* 2013, 2013:3 http://www.fixedpointtheoryandapplications.com/content/2013/1/3