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Truncated Rayleigh Lomax Distribution

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Abstract:- **The Truncated Rayleigh - Lomax distribution is introduced in this research. To do this, we interrupt the period of Rayleigh-Lomax to create a distribution with more flexibility and efficiency. Then we extracted the statistical and mathematical characters of the distribution, like, cumulative distribution, survival, density, Hazard, and cumulative Hazard functions. We also found the arithmetic mean, median, mode, order statistics, and moment-generating function extracted the Kurtosis and Skewedness, Variation Coefficients, and presented some analysis methods. Finally, we compared the new truncated Rayleigh-Lomax distribution with the original Rayleigh-Lomax distributions and some other distributions. The results showed that the truncated Rayleigh -Lomax distributions was better than the Rayleigh -Lomax distribution.**

Keywords: Cumulative function, density, Reliability, moment generating function, reverse hazard, hazard, cumulative hazard function, mean and variance, mode, median, quintile, order statistics, moments, truncated Rayleigh Lomax distribution.

INTRODUCTION

We are sometimes required by design that is forced to truncate some of the distribution's observations or delete them to take advantage of the time. In this case, only the sample taken from the truncated distribution is used for estimation purposes, and removing part of the possible values for this distribution, is done in one or two parts. Generally, any value outside the period $[a, b]$ is ignored.

Since $f_T(x;\theta)$ represents the truncated probability distribution function. That is, this function fulfills the conditions of the probability density function, i.e. $f_T \geq 0$ and $\int_a^b f_T(X;\theta)dx = 1$ or $(\sum_a^b P_T\,(X = x_0;\theta) = 1)$. It is important to remember that the truncation process changes the original distributions' mean, variance, and other statistical measures.

The goal of discovering or developing the distribution is new because of its importance, usefulness, and practical and scientific reality in several areas of life, such as biostatistics or the analysis of survival function. So in this research, a new distribution was found: the truncate Rayleigh-Lomax distribution. This is done by truncating the period for the distribution of Rayleigh-Lomax.

1. Truncated Rayleigh – Lomax distribution.

The distribution in which the period truncated at point $a_\circ$, in which $a_\circ$ is a constant. i.e., All of the values chosen at random are in the range $0 < x < a_\circ$.

2. PDF and CDF of TRLD

The pdf of the (TRLD) is given by

$$g(x;\beta,b,\theta) = \frac{f(x;\beta,b,\theta)}{F(b)-F(a)} = \frac{\frac{\theta\beta}{2b^2}(1+\beta x)^{\theta-1}\, e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-\, e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}, \tag{1}$$

$\frac{-1}{\beta} \leq x \leq a_\circ$ & $\beta, b, \theta > 0$

Such as $\int_{\frac{-1}{\beta}}^{a_\circ} \frac{\frac{\theta\beta}{2b^2}(1+\beta x)^{\theta-1} e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} = 1$

For the (TRLD), the corresponding plot to pdf is as follows:

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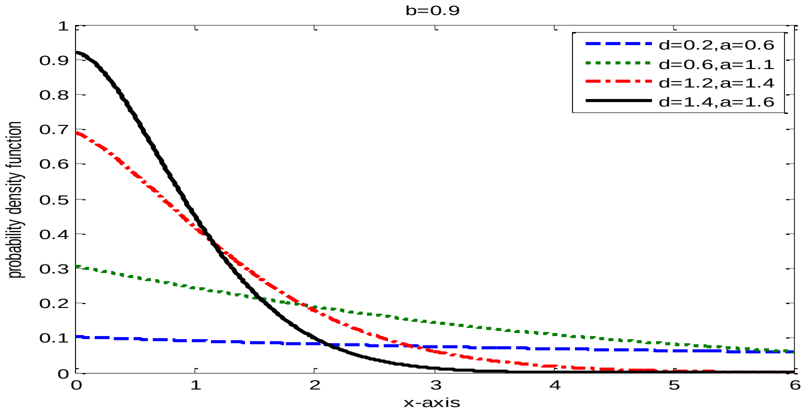

Figure.1: p.d.f plotting of (TRLD), parameters b=0.9, theta=0.6,1.1,1.4,1.6 ; d=0.2,0.6,1.2,1.4

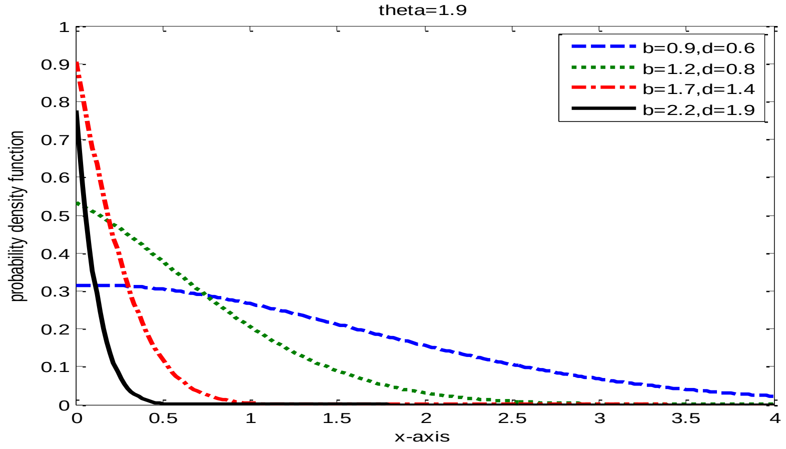

Figure.2: p.d.f plotting of (TRLD), parameters theta=1.9, b=0.9,1.2,1.7,2.2 ; d=0.6,0.8,1.4,1.9

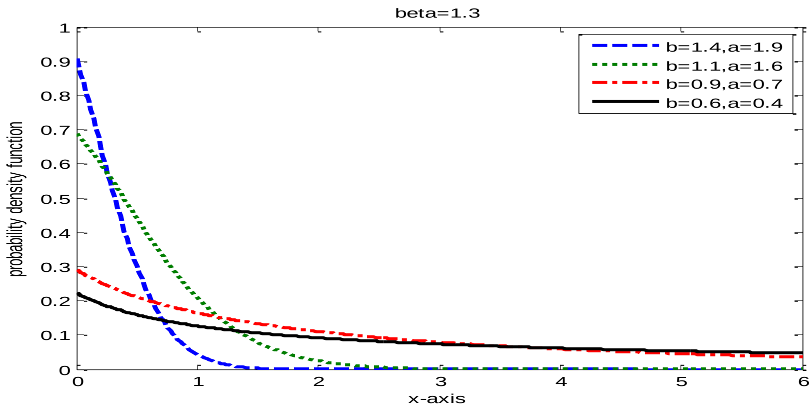

Figure.3: p.d.f plotting of (TRLD), parameters beta=1.3, b=1.4,1.1,0.9,0.6 ; a=1.9,1.6,0.7,0.4

Figures (1), (2) and (3) indicate that the TRPD family generates various shapes such as symmetrical, left – skewed, right skewed and reversed- J

$$G(x;\beta,b,\theta)=\int_{\frac{-1}{\beta}}^{x} g(x;\beta,b,\theta)=\int_{\frac{-1}{\beta}}^{x}\frac{\frac{\beta\theta}{2b^2}\,(1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-\,e^{-\frac{1}{2b^2}(1+\beta a_{\circ})^{\theta}}}$$

$$=\frac{1-\,e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-\,e^{-\frac{1}{2b^2}(1+\beta a_{\circ})^{\theta}}},\frac{-1}{\beta}\le x\le a_{\circ}\ \&\ \beta,b,\theta>0 \qquad (2)$$

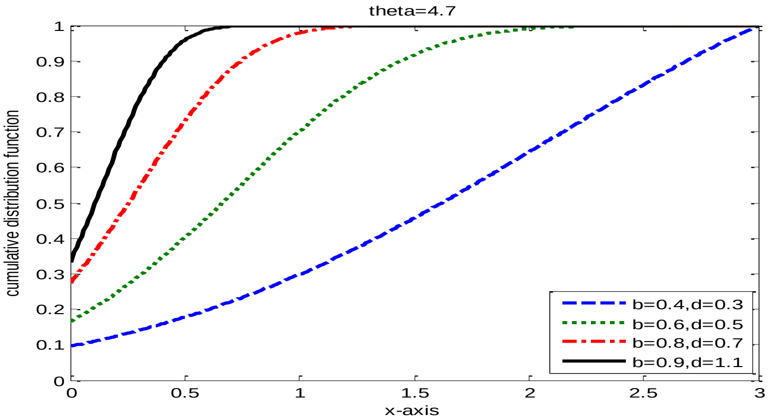

Figure.4: c.d.f plotting of (TRLD), parameters theta=4.7, b=0.4,0.6,0.8,0.9 ;d =0.3,0.5,0.7,1.1

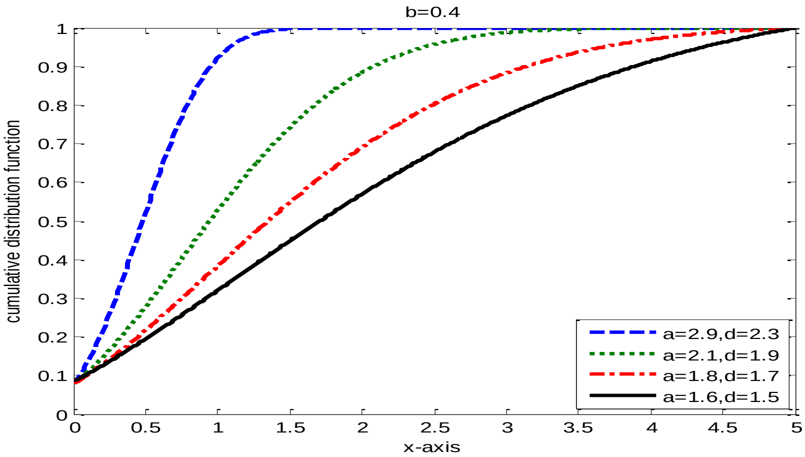

Figure.5: c.d.f plotting of (TRLD), parameters b=0.4, a=2.9,2.1,1.8,1.6 ; d =2.3,1.9,1.7,1.5

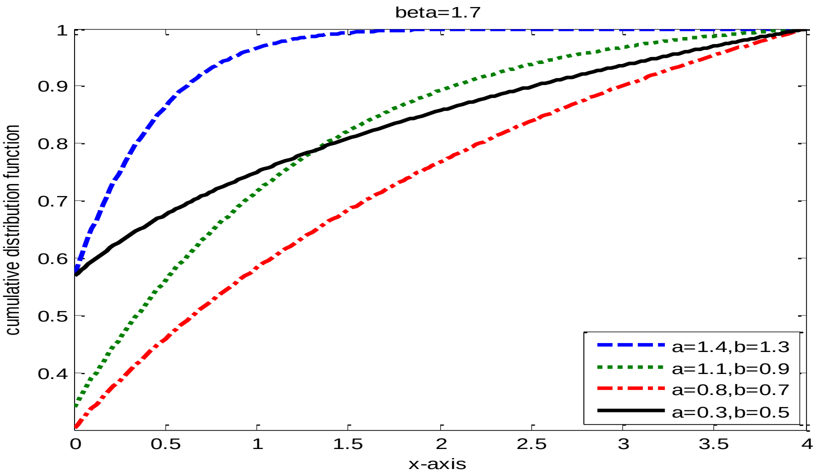

Figure.6: c.d.f plotting of (TRLD), parameters beta=1.7, a=1.4,1.1,0.8,0.3 ; b=1.3,0.9,0.7,0.5

Figures (4), (5), and (6) show that when x and the parameters $\alpha, w, and\ q$ are increased, the cdf of the TRPD does not decrease.

Figures (4), (5), and (6) demonstrate that the cdf of the TRPD does not decrease when x and the parameters, $\alpha, w, and\ q$ are increased.

3. Limitation of c.d.f and p.d.f

The distribution's limitations are determined via:

$$\lim_{x\to\frac{-1}{\beta}} g(x;\beta,b,\theta) = 0 \qquad (3)$$

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$$\lim_{x\to\frac{-1}{\beta}} \frac{\frac{\beta\theta}{2b^2}(1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} = 0$$

Also,

$$\lim_{x\to a_\circ} g(x;\beta,b,\theta) = \lim_{x\to a_\circ} \frac{\frac{\beta\theta}{2b^2}(1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} = \frac{\frac{\beta\theta}{2b^2}(1+\beta a_\circ)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} > 0 \quad (4)$$

Because $1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}} > 0$ and

$\frac{\beta\theta}{2b^2}(1+\beta a_\circ)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}} > 0$

As a result, this distribution's c.d.f. is as follows:

$$\lim_{x\to\frac{-1}{\beta}} G(x;\beta,b,\theta) = \lim_{x\to\frac{-1}{\beta}} \frac{1-e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} = 0 \quad (5)$$

Also**,**

$$\lim_{x\to a_\circ} G(x;\beta,b,\theta) = \lim_{x\to a_\circ} \frac{1-e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} = 1 \quad (6)$$

i.e $0 \le G(x;\beta,b,\theta) \le 1$

4. Some Survival Functions

In this section, some survival functions for the TRLD will be introduced.

4.1. Survival Function

The survival function of TRLD is given by:

$$S(x) = 1 - G(x) = 1 - \frac{1-e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} = \frac{e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}} - e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} \quad (7)$$

Such as,

$$\lim_{x\to\frac{-1}{\beta}} S(x) = \lim_{x\to\frac{-1}{\beta}} \frac{e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}} - e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} = 1$$

$$\lim_{x\to a_\circ} S(x) = \lim_{x\to a_\circ} \frac{e^{-\frac{1}{2b^2}(1+\beta x)^{\alpha}} - e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} = 0$$

That is 1) $G(X) \to 0, and\ S(X) \to 1\ as\ X \to \frac{-1}{\beta}$

2) $G(X) \to 1, and\ S(X) \to 0\ as\ X \to 0$

The plot of the $S(x)$ and the $h(x)$ for the(TRLD) as follows.

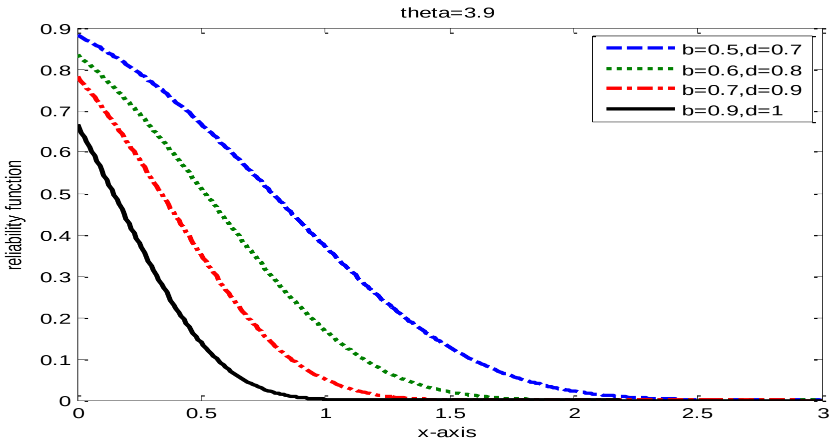

Figure.7: Plotting of $S(X)$ for (TRLD), parameters theta=3.9, b=0.5,0.6,0.7,0.9 ; d=0.7,0.8,0.9,1

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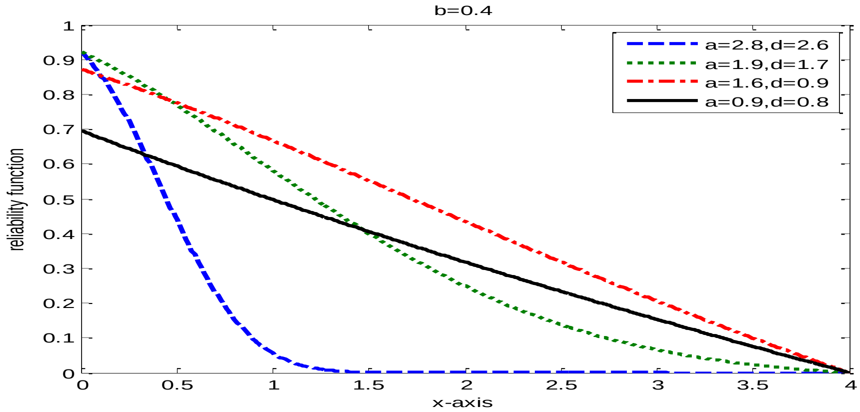

Figure.8: Plotting of $S(X)$ for (TRLD), parameters b=3.9, a=2.8,1.9,1.6,0.9 ; d=2.6,1.7,0.9,0.8

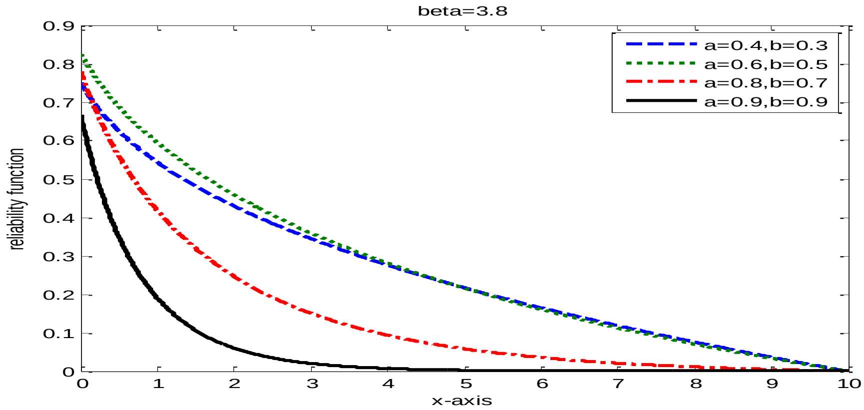

Figure .9: Plotting of $S(X)$ for (TRLD), parameters beta=3.8, a=0.4,0.6,0.8,0.9 ; b=0.3,0.5,0.7,0.9

The $S(x)$ of the TRPD is shown in Figures (7), (8), and (9) to be a decreasing function.

4.2. Hazard Function

The hazard function for TRLD is given by:

$$h(x) = \frac{g(x;\beta,b,\alpha)}{S(x)} = \frac{\frac{\beta\theta}{2b^2}\,(1+\beta x)^{\theta-1} e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}} - e^{-\frac{1}{2b^2}(1+\beta a_{\circ})^{\theta}}} \quad (8)$$

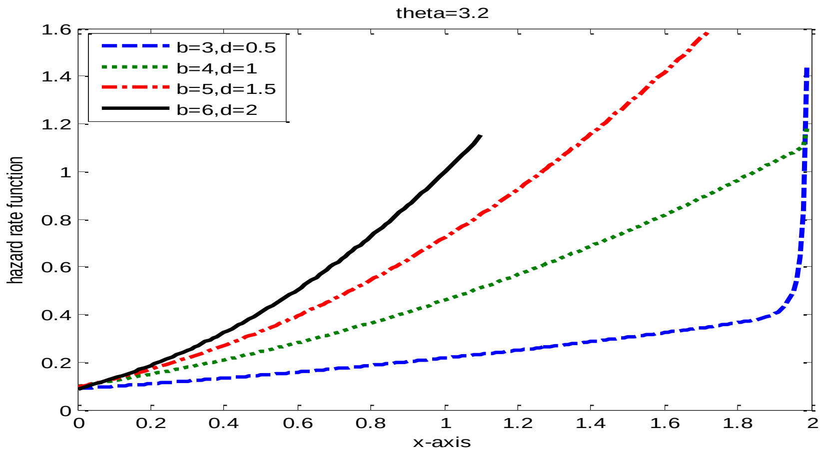

Figure.10: Plotting of $h(X)$ for (TRLD), parameter theta=3.2, b=3,4,5,6 ; d=0.5,1,1.5,2

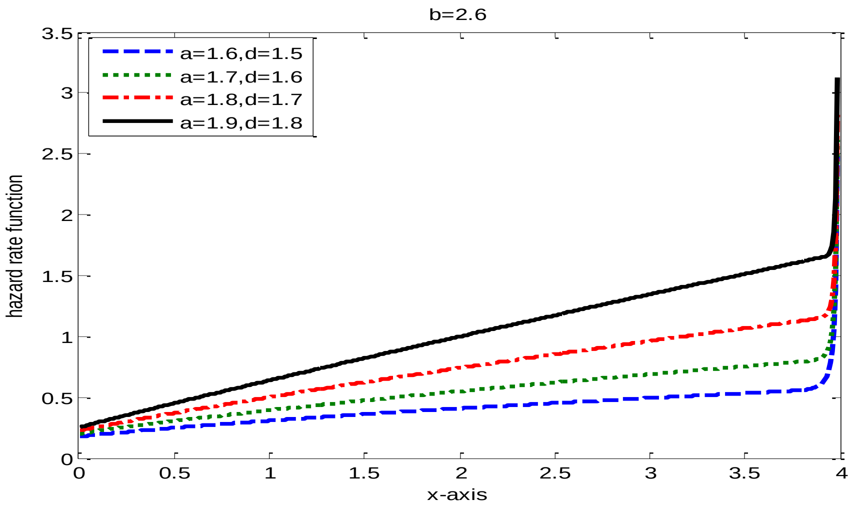

Figure.11: Plotting of $h(x)$of the (TRLD) for parameter b=2.6, a=1.6,1.7,1.8,1.9 ; d=1.5,1.6,1.7,1.8

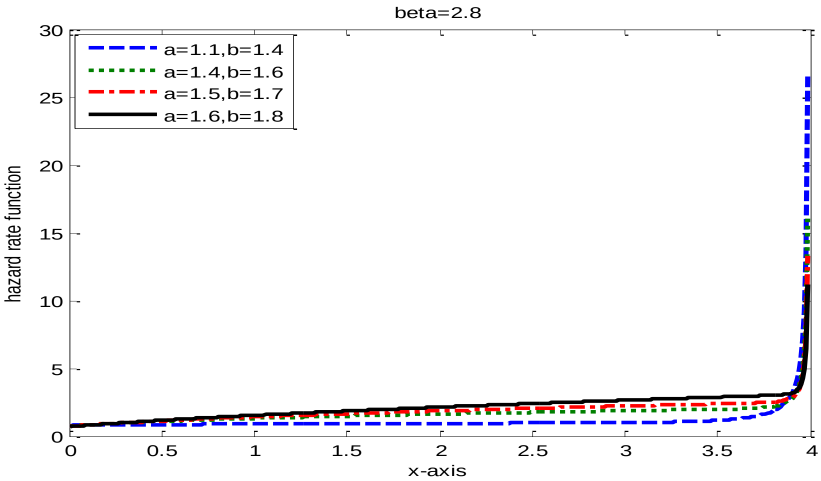

Figure.12: Plotting of $h(X)$ for (TRLD), parameter beta=2.8, a=1.1,1.4,1.5,1.6 ; b=1.4,1.6,1.7,1.8.

4.3. Reverse Hazard Function

The reverse hazard function for TRLD is:

$$r(x) = \frac{g(x;\beta,b,\theta)}{G(x;\beta,b,\theta)} = \frac{\frac{\beta\theta}{2b^2}\,(1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-\ e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}} \qquad (9)$$

4.4. The Cumulative Hazard Function

The formulation for the TRLD cumulative hazard function is:

$$H(x) = \ -ln\ S(x) = \ -ln\,\frac{e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}} - e^{-\frac{1}{2b^2}(1+\beta a_{\circ})^{\theta}}}{1-\ e^{-\frac{1}{2b^2}(1+\beta a_{\circ})^{\theta}}} \qquad (10)$$

Such as, $x,\beta,b,\theta > 0$

5.Some properties of the TRLD :

5.1 mode:

Proposition1:

The mode of the TRLD is.

$$x_{mode} = \frac{(\frac{2b^2(\theta-1)}{\theta})^{\frac{1}{\theta}} - 1}{\beta} \qquad (11)$$

Proof:

$$x_{mode} = \frac{dln\ g(x_{mode},\beta,b,\theta)}{dx} = 0 \qquad (12)$$

$$\frac{dln\ g(x_{mode},\beta,b,\theta)}{\partial x}=\frac{\partial}{\partial x}\left[ln\frac{\frac{\beta\theta}{2b^2}\ (1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}\right]=0$$

$$\Rightarrow \frac{\partial}{\partial x}[\ ln\beta+ln\theta-ln2b^2+(\theta-1)ln(1+\beta x)-\frac{1}{2b^2}(1+\beta x)^{\theta}-ln\left(1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right)]=0$$

$$\Rightarrow (\theta-1)\frac{\beta}{(1+\beta x_{mode})}-\frac{\theta\beta}{2b^2}(1+\beta x_{mode})^{\theta-1}=0\Rightarrow$$

$$(\theta-1)\frac{\beta}{(1+\beta x_{mode})}=\frac{\theta\beta}{2b^2}(1+\beta x_{mode})^{\theta-1}$$

$$x_{mode}=\frac{(\frac{2b^2(\theta-1)}{\theta})^{\frac{1}{\theta}}-1}{\beta} \qquad (13)$$

5.2 Quintile and Median:

Proposition 2

For the TRLD, the formula for the quintile x_q and the median is as follows:

$$x_q=\frac{[-2b^2\ ln\ (\ 1-q\left[1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right]]^{\frac{1}{\theta}}-1}{\beta} \qquad (14)$$

$$x_{median}=\frac{[-2b^2\ ln\ (\frac{1}{2}\left[1+\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right]]^{\frac{1}{\theta}}-1}{\beta} \qquad (15)$$

Proof:

From equation (3), obtain.

$$\frac{1-\ e^{-\frac{1}{2b^2}(1+\beta x_q)^{\theta}}}{1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\alpha}}}=q\ \Rightarrow\ 1-\ e^{-\frac{1}{2b^2}(1+\beta x_q)^{\theta}}=q\left[1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right]$$

$$\Rightarrow e^{-\frac{1}{2b^2}(1+\beta x_q)^{\theta}}=1-\ q\left[1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right]$$

Take Ln to the two parties

$$-\frac{1}{2b^2}(1+\beta x_q)^{\theta}=ln\left(1-\ q\left[1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right]\right)$$

Then,

$$x_q=\frac{[-2b^2\ ln\ (\ 1-q\left[1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right])]^{\frac{1}{\theta}}-1}{\beta} \qquad (16)$$

Changing $q=\frac{1}{2}$ in equation (**16**) for TRLD's median yields:

$$x_{median}=\frac{[-2b^2\ ln\ (\ 1-\frac{1}{2}\left[1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right])]^{\frac{1}{\theta}}-1}{\beta}\Rightarrow$$

$$x_{median}=\frac{[-2b^2\ ln\ (\frac{1}{2}\left[1+\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right]]^{\frac{1}{\theta}}-1}{\beta} \qquad (17)$$

5.3.Moment Generation Function:

Proposition3

The TRLD moment generation function is the formula:

$$M_x(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\binom{j\theta}{k}\binom{\theta-1}{r}(\beta)^{(j+1)\theta+i-r-k}$$
$$\times\frac{t^j(-1)^j\theta[(a_\circ)^{(j+1)\theta+i-r-k}-(\frac{-1}{\beta})^{(j+1)\theta+i-r-k}]}{(j!)(i!)(2b^2)^{j+1}(1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}})((j+1)\theta+i-r-k)} \qquad (18)$$

Proof:

$$M_x(t) = E(e^{xt}) = \int_{\frac{-1}{\beta}}^{a_\circ} e^{xt} \frac{\frac{\beta\theta}{2b^2}(1+\beta x)^{\theta-1} e^{-\frac{1}{2b^2}(1+\beta x)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}} dx$$

By using series expansion of e^{xt} get $e^{tx} = \sum_{i=0}^{\infty} \frac{t^i x^i}{i!}$ So,

$$M_x(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \binom{j\theta}{k}\binom{\theta-1}{r} (\beta)^{(j+1)\theta-r-k} \frac{t^i(-1)^j\theta}{(j!)(i!)(2b^2)^{j+1}(1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta})} \times \int_{\frac{-1}{\beta}}^{a_\circ} x^{(j+1)\theta+i-r-k-1}\, dx$$

Then,

$$M_x(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \binom{j\theta}{k}\binom{\theta-1}{r} (\beta)^{(j+1)\theta+i-r-k} \times \frac{t^j(-1)^j\theta[(a_\circ)^{(j+1)\theta+i-r-k} - (\frac{-1}{\beta})^{(j+1)\theta+i-r-k}]}{(j!)(i!)(2b^2)^{j+1}(1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta})((j+1)\theta+i-r-k)} \quad (19)$$

6. Moments

Theorem1:

The TRLD's r^{th} moment with respect to the origin and r^{th} moment with respect to the mean are as follows:

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty} \binom{j\theta}{i}\binom{\theta-1}{h} (\beta)^{(j+1)\theta-i-h} \times \frac{(-1)^j\theta[(a_\circ)^{(j+1)\theta-i-h+r+1} - (\frac{-1}{\beta})(j+1)^{(j+1)\theta-i-h+r}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]\,((j+1)\theta-i-h+r)} \quad (20)$$

$$E(X-\mu)^r = \sum_{n=0}^{r}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} \binom{r}{n}\binom{j\theta}{k}\binom{\theta-1}{h} (-\mu)^{r-n} (\beta)^{(j+1)\theta-h-k} \times \frac{(-1)^j\theta[(a_\circ)^{(j+1)\theta-h-k+n+1} - (\frac{-1}{\beta})^{(j+1)\theta-h-k+n+1}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]\,((j+1)\theta-h-k+n+1)} \quad (21)$$

r=1,2,3,….n

Proof: Regarding the origin, the r^{th} is:

$$E(X^r) = \int_{\frac{-1}{\beta}}^{a_\circ} X^r g(x;p,b,\alpha) d = \int_{\frac{-1}{\beta}}^{a_\circ} X^r \frac{\frac{\theta\beta}{2b^2}(1+\beta x)^{\theta-1} e^{-\frac{1}{2b^2}(1+\beta x)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}} dx$$

$$= \frac{\theta\beta}{2b^2[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]} \int_{\frac{-1}{\beta}}^{a_\circ} X^r (1+\beta x)^{\theta-1} e^{-\frac{1}{2b^2}(1+\beta x)^\theta} dx$$

Then,

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty} \binom{j\theta}{i}\binom{\theta-1}{h} (\beta)^{(j+1)\theta-i-h} \frac{(-1)^j\theta}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]} \times \int_{\frac{-1}{\beta}}^{a_\circ} x^{(j+1)\theta-i-h+r-1} dx$$

Then,

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\binom{j\theta}{i}\binom{\theta-1}{h}(\beta)^{(j+1)\theta-i-h} \times \frac{(-1)^j\theta[(a_\circ)^{(j+1)\theta-i-h+r} - (\frac{-1}{\beta})^{(j+1)\theta-i-h+r}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]\,((j+1)\theta-i-h+r)} \tag{22}$$

when $r = 1,2$

$$E(X) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\binom{j\theta}{i}\binom{\theta-1}{h}(\beta)^{(j+1)\theta-i-h} \times \frac{(-1)^j\theta[(a_\circ)^{(j+1)\theta-i-h+1} - (\frac{-1}{\beta})^{(j+1)\theta-i-h+1}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]\,((j+1)\theta-i-h+1)} \tag{23}$$

$$E(X^2) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\binom{j\theta}{i}\binom{\theta-1}{h}(\beta)^{(j+1)\theta-i-h} \times \frac{(-1)^j\theta[(a_\circ)^{(j+1)\theta-i-h+2} - (\frac{-1}{\beta})^{(j+1)\theta-i-h+2}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]\,((j+1)\theta-i-h+2)} \tag{24}$$

$$var(X) = E(X^2) - [E(X)]^2 \tag{25}$$

The mean's rth moment is provided via:

$$E(X-\mu)^r = \int_{\frac{-1}{\beta}}^{a_\circ} (X-\mu)^r \frac{\frac{\theta\beta}{2b^2}(1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}}dx$$

Where

$$(X-\mu)^r = \sum_{n=0}^{r}\binom{r}{n}(X)^n(-\mu)^{r-n}$$

Therefore, mean's rth moment is provided via:

$$E(X-\mu)^r = \sum_{n=0}^{r}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\binom{r}{n}\binom{j\theta}{k}\binom{\theta-1}{h}(-\mu)^{r-n}(\beta)^{(j+1)\theta-h-k} \times \frac{(-1)^j\theta[(a_\circ)^{(j+1)\theta-h-k+n+1} - (\frac{-1}{\beta})^{(j+1)\theta-h-k+n}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]\,((j+1)\theta-h-k+n)} \tag{26}$$

7. Order Statistics

Suppose that $x_1, x_2, \ldots, x_n$ denoted a r.s for the size n from a TRLD with $g(x;\beta,b,\theta)$ and $G(x;\beta,b,\theta)$ in the equation (1) and (2) . Let $Y_{k1}, Y_{k2}, \ldots, Y_{kn}$ denote the correlating orders statistical; so, the p.d.f of $X_{k:n}$ is provided via:

$$g_{k,n}(x,\beta,b,\theta) = \frac{n!}{(k-1)!(n-k)!}g(x;\beta,b,\theta)[G(x;\beta,b,\theta)]^{k-1}[1-G(x;\beta,b,\theta)]^{n-k}$$

The r^{th} order statistics p.d.f. is derived from the TRLD's p.d.f.

$$g_{k,n}(x,\beta,b,\theta) = \frac{n!}{(k-1)!\,(n-k)!}\left[\frac{\frac{\beta\theta}{2b^2}(1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}}\right] \times \left[\frac{1-e^{-\frac{1}{2b^2}(1+\beta x)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}}\right]^{k-1}\left[\frac{e^{-\frac{1}{2b^2}(1+\beta x)^\theta}-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}}\right]^{n-k} \tag{27}$$

So, we can define the median, maximum, and minimum p.d.f. as follows:
1- p.d.f of minimum if k=1:

$$g_{1,n}(x,\beta,b,\theta) = \frac{n\frac{\beta\theta}{2b^2}(1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a)^{\theta}}}\left[\frac{e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}\right]^{n-1} \tag{28}$$

2- p.d.f of maximum if k=n:

$$g_{n,n}(x,\beta,b,\theta) = \frac{n\frac{\beta\theta}{2b^2}(1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}\left[\frac{1-e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}\right]^{n-1} \tag{29}$$

3- p.d.f of median if k=m+1:

$$g_{m+1,n}(x,\beta,b,\theta) = \frac{n!}{m!\,(n-m-1)!}\left[\frac{\frac{\beta\theta}{2b^2}(1+\beta x)^{\theta}e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}\right]\left[\frac{1-e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}\right]^{m}\left[1-\frac{1-e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}\right]^{n-m-1} \tag{30}$$

8. Skewedness, Kurtosis, and Variation Coefficients

In this section, we introduce and study Skewedness, kurtosis and variation of (TRLD) based on the moment as the following proposition.

Proposition 4

The, variation coefficients, skewedness-kurtosis of the TRLD are presented via:

$$CS = \frac{E(X-\mu)^3}{\sigma^3} \tag{31}$$

Let $CS = \frac{A}{B}$

By equation (26), get

$$A = \sum_{n=0}^{3}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\binom{3}{n}\binom{j\theta}{k}\binom{\theta-1}{h}(-\mu)^{3-n}(\beta)^{(j+1)\theta-h-k} \times \frac{(-1)^j\theta[(a_\circ)^{(j+1)\theta-h-k+n+1}-(\frac{-1}{\beta})^{(j+1)\theta-h-k+n}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}]\,((j+1)\theta-h-k+n)}$$

$$B = \left[\sum_{n=0}^{2}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\binom{r}{n}\binom{j\theta}{k}\binom{\theta-1}{h}(-\mu)^{2-n}(\beta)^{(j+1)\theta-h-k} \times \frac{(-1)^j\theta[(a_\circ)^{(j+1)\theta-h-k+n+1}-(\frac{-1}{\beta})^{(j+1)\theta-h-k+n}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}]\,((j+1)\theta-h-k+n)}\right]^{\frac{3}{2}}$$

$$CK = \frac{E(X-\mu)^4}{\sigma^4} \tag{32}$$

Let $CK = \frac{C}{D}$

By equation (26), get

$$C = \sum_{n=0}^{4}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\binom{4}{n}\binom{j\theta}{k}\binom{\theta-1}{h}(-\mu)^{4-n}(\beta)^{(j+1)\theta-h-k} \times \frac{(-1)^j\theta[(a_\circ)^{(j+1)\theta-h-k+n+1}-(\frac{-1}{\beta})^{(j+1)\theta-h-k+n+1}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}]\,((j+1)\theta-h-k+n+1)}$$

$$D = \left[\sum_{n=0}^{2}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\binom{2}{n}\binom{j\theta}{k}\binom{\theta-1}{h}(-\mu)^{2-n}(\beta)^{(j+1)\theta-h-k}\right.$$

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$$\times \frac{(-1)^j \theta [(a_\circ)^{(j+1)\theta-h-k+n+1} - (\frac{-1}{\beta})^{(j+1)\theta-h-k+n+1}]}{j!\,(2b^2)^{j+1}[1- e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]\,((j+1)\theta-h-k+n+1)}]^2$$

$$CV = \frac{\sigma}{\mu} \tag{33}$$

Let $CV = \frac{E}{F}$

$$E = [\sum_{n=0}^{2}\sum_{j=0}^{\infty}\ \sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\binom{2}{n}\binom{j\theta}{k}\binom{\theta-1}{h}(-\mu)^{2-n}(\beta)^{(j+1)\theta-h-k}$$

$$\times \frac{(-1)^j \theta [(a_\circ)^{(j+1)\theta-h-k+n+1} - (\frac{-1}{\beta})^{(j+1)\theta-h-k+n+1}]}{j!\,(2b^2)^{j+1}[1- e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]\,((j+1)\theta-h-k+n+1)}]^{\frac{1}{2}}$$

$$F = \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\binom{j\theta}{k}\binom{\theta-1}{r}(\beta)^{(j+1)\theta-r-k}$$

$$\times \frac{(-1)^j \theta [(a_\circ)^{(j+1)\theta-r-k+1} - (\frac{-1}{\beta})^{(j+1)\theta-r-k+1}]}{j!\,(2b^2)^{j+1}[1- e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]\,((j+1)\theta-r-k+1)}$$

Proposition 5:

The harmonic mean is given by:

$$H = E\left(\frac{1}{X}\right) = \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{j\theta}{s}\binom{\theta-1}{r}$$

$$\times \frac{\beta^{(j+1)\theta-r-s}(-1)^j[(a_\circ)^{(j+1)\theta-r-s-1} - (\frac{-1}{\beta})^{(j+1)\theta-r-s-1}]}{j!\,(2b^2)^{j+1}[1- e^{-\frac{1}{2b^2}(1+\beta a)^\theta}]((j+1)\theta-r-s-1)} \tag{34}$$

Proof:

$$H = E\left(\frac{1}{X}\right) = \int_{\frac{-1}{\beta}}^{a_\circ}\frac{1}{X}\,g(x;\beta,b,\theta)\partial x = \int_{\frac{-1}{\beta}}^{a_\circ}\frac{1}{X}\,\frac{\frac{\theta\beta}{2b^2}(1+\beta x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\beta x)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a)^\theta}}dx$$

$$= \frac{\theta\beta}{2b^2[1-e^{-\frac{1}{2b^2}(1+\beta a)^\theta}]}\int_{\frac{-1}{\beta}}^{a_\circ}X^{-1}(1+\beta x)^{\theta-1}\,e^{-\frac{1}{2b^2}(1+\beta x)^\theta}dx$$

Thus,

$$E\left(\frac{1}{X}\right) = \frac{\theta}{2b^2\left[1- e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}\right]}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{j\theta}{s}\binom{\theta-1}{r}(\beta)^{(j+1)\theta-r-s}$$

$$\times \frac{(-\frac{1}{2b^2})^j}{j!}\int_{\frac{-1}{\beta}}^{a_\circ}x^{(j+1)\theta-r-s-2}dx$$

Thus,

$$E\left(\frac{1}{X}\right) = \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{j\theta}{s}\binom{\theta-1}{r}$$

$$\times \frac{\beta^{(j+1)\theta-r-s}(-1)^j[(a_\circ)^{(j+1)\theta-r-s-1} - (\frac{-1}{\beta})^{(j+1)\theta-r-s-1}]}{j!\,(2b^2)^{j+1}[1- e^{-\frac{1}{2b^2}(1+\beta a)^\theta}]((j+1)\theta-r-s-1)} \tag{35}$$

Proposition 6:

The formula for the geometrical mean is:

$$G = E(\sqrt{X}) = \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{j\theta}{s}\binom{\theta-1}{r} \times \frac{\beta^{(j+1)\theta-r-s}(-1)^j[(a_\circ)^{(j+1)\theta-r-s-1} - (\frac{-1}{\beta})^{(j+1)\theta-r-s+\frac{1}{2}}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a)^\theta}]((j+1)\theta-r-s+\frac{1}{2})} \quad (36)$$

Proof:

$$G = E(\sqrt{X}) = \int_{\frac{-1}{\beta}}^{a_\circ} \sqrt{X}\, g(x;\beta,b,\theta)dx$$

$$= \int_{\frac{-1}{\beta}}^{a_\circ} X^{\frac{1}{2}} \frac{\frac{\theta\beta}{2b^2}(1+\beta x)^{\theta-1} e^{-\frac{1}{2b^2}(1+\beta x)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a)^\theta}}dx = \frac{\theta\beta}{2b^2[1-e^{-\frac{1}{2b^2}(1+\beta a)^\theta}]}\int_{\frac{-1}{\beta}}^{a_\circ} X^{\frac{1}{2}}(1+\beta x)^{\theta-1} e^{-\frac{1}{2b^2}(1+\beta x)^\theta}dx$$

Then,

$$E(\sqrt{X}) = \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{j\theta}{s}\binom{\theta-1}{r} \times \frac{\beta^{(j+1)\theta-r-s}(-1)^j[(a_\circ)^{(j+1)\theta-r-s-1} - (\frac{-1}{\beta})^{(j+1)\theta-r-s+\frac{1}{2}}]}{j!\,(2b^2)^{j+1}[1-e^{-\frac{1}{2b^2}(1+\beta a)^\theta}]((j+1)\theta-r-s+\frac{1}{2})} \quad (37)$$

***9. Renyi Entropy*:** Following is the Renyi entropy notation of the p.d.f. random variable x:

$$H_R(\delta) = \frac{1}{1-\delta} log \int_{\frac{-1}{\beta}}^{a_\circ} f^{\delta}(x)dx\,, \qquad where\ \delta > 0\ \&\ \delta \neq 1 \quad (38)$$

Proposition 7:

The Renyi entropy of x, when x is a random variable has a TRPD.

$$H_R(\delta) = \frac{1}{1-\delta} log[\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\binom{j\theta}{i}\binom{\theta\delta-\delta}{h}(\beta)^{(j+\delta)\theta-i-h} \times \frac{(-1)^j\theta^\delta\delta^j[(a_\circ)^{(j+\delta)\theta-\delta-i-h+1} - (\frac{-1}{\beta})^{(j+\delta)\theta-\delta-i-h+1}]}{j!\,(2b^2)^{j+\delta}[1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}]^\delta\,((j+\delta)\theta-\delta-i-h+1)}] \quad (39)$$

Proof:

$$H_R(\delta) = \frac{1}{1-\delta}\log\left[\int_{-\frac{1}{\beta}}^{a_\circ} \frac{\frac{\theta\beta}{2b^2}(1+\beta x)^{\theta-1} e^{-\frac{1}{2b^2}(1+\beta x)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}}\right]dx \quad (40)$$

By integrating equation (40), get equation (39) by using the same steps in $E(X^r)$.

10. Estimation Methods

There will be a discussion of several approaches for estimating TRLD's unknown parameters.

10.1. Maximum Likelihood Estimation.

Maximum likelihood estimation is employed to estimate the TRLD's unknown parameters. The p.d.f. the likelihood function is defined as follows:

$$L(x,\beta,b,\theta) = \frac{\frac{\beta^n\theta^n}{(2b^2)^n}\prod_{i=1}^{n}(1+\beta x)^{\theta-1} e^{-\frac{1}{2b^2}\sum_{i=1}^{n}(1+\beta x_i)^\theta}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^\theta}}$$

The likelihood function with the Ln of both sides is taken.

$$\ell = n ln\,\beta + n ln\theta - n ln(2b^2) + (\theta-1)\sum_{i=1}^{n} ln(1+\beta x_i) - \frac{1}{2b^2}\sum_{i=1}^{n}(1+\beta x_i)^{\theta} - ln\left(1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}\right) \quad (41)$$

Now, derive the equation (3.43) with respect to β, b and θ

$$\frac{\partial Ln\ell}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} ln(1+\beta x_i) - \frac{1}{2b^2}\sum_{i=1}^{n}(1+\beta x_i)^{\theta} ln(1+\beta x_i) - \frac{\frac{1}{2b^2}(1+\beta a_\circ)^{\theta} ln(1+\beta a_\circ) e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} \quad (42)$$

$$\frac{\partial Ln\ell}{\partial b} = \frac{-2n}{b} + \frac{1}{b^3}\sum_{i=1}^{n}(1+\beta x)^{\theta} - \frac{\frac{1}{b^3}(1+\beta a_\circ)^{\theta} e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} \quad (43)$$

$$\frac{\partial Ln\ell}{\partial\beta} = \frac{n}{\beta} - \frac{(\theta-1)\sum_{i=1}^{n} x_i}{(1+\beta\sum_{i=1}^{n} x_i)} - \frac{\theta\sum_{i=1}^{n}(1+\beta x)^{\theta-1}}{2b^2} - \frac{\frac{\theta a_\circ}{2b^2}(1+\beta a_\circ)^{\theta-1} e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}} \quad (44)$$

Lastly, set every one of these equations equal to 0.

$$\frac{n}{\hat\theta} + \sum_{i=1}^{n} ln(1+\hat\beta x_i) - \frac{1}{2\hat b^2}\sum_{i=1}^{n}(1+\hat\beta x_i)^{\hat\theta} ln(1+\hat\beta x_i) - \frac{\frac{1}{2\hat b^2}(1+\hat\beta a_\circ)^{\hat\theta} ln(1+\hat\beta a_\circ) e^{-\frac{1}{2\hat b^2}(1+\hat\beta a_\circ)^{\hat\theta}}}{1-e^{-\frac{1}{2\hat b^2}(1+\hat\beta a_\circ)^{\hat\theta}}} = 0 \quad (45)$$

$$\frac{-2n}{\hat b} + \frac{1}{\hat b^3}\sum_{i=1}^{n}(1+\hat\beta x)^{\hat\theta} - \frac{\frac{1}{\hat b^3}(1+\hat\beta a_\circ)^{\hat\theta} e^{-\frac{1}{2\hat b^2}(1+\hat\beta a_\circ)^{\hat\theta}}}{1-e^{-\frac{1}{2\hat b^2}(1+\hat\beta a_\circ)^{\hat\theta}}} = 0 \quad (46)$$

$$\frac{n}{\hat\beta} - \frac{(\hat\theta-1)\sum_{i=1}^{n} x_i}{(1+\hat\beta\sum_{i=1}^{n} x_i)} - \frac{\hat\theta\sum_{i=1}^{n}(1+\hat\beta x)^{\hat\theta-1}}{2\hat b^2} - \frac{\frac{\hat\theta a_\circ}{2\hat b^2}(1+\hat\beta a)^{\hat\theta-1} e^{-\frac{1}{2\hat b^2}(1+\hat\beta a_\circ)^{\hat\theta}}}{1-e^{-\frac{1}{2\hat b^2}(1+\hat\beta a_\circ)^{\hat\theta}}} = 0 \quad (47)$$

In order to determine the MLEs of the parameters β, b and θ and, we solve equations (45) through (47) by using numerical techniques.

10.2. Least Square Method (LS):

Following is a working definition of the method. It is well known that.

$E\left(G\left(Y_{(i)}\right)\right) = \frac{i}{n+1}$

Obtain the estimators by minimizing.

$\sum_{i=1}^{n}(G\left(Y_{(i)}\right) - \frac{i}{n+1})^2$

SO

$$\sum_{i=1}^{n}[\left(\frac{1-\ e^{-\frac{1}{2b^2}(1+\beta x)^{\theta}}}{1-\ e^{-\frac{1}{2b^2}(1+\beta a_\circ)^{\theta}}}\right) - \frac{i}{n+1}]^2 \quad (48)$$

The $\hat b_{LSE}$ could exist by deriving equation (48) concerning b.

$\sum_{i=1}^{n} 2[\ (\frac{1-e^{-\frac{1}{2\hat b^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2\hat b^2}(1+\beta a_\circ)^{\theta}}}) - \frac{i}{n+1}] \times[\frac{(1-e^{-\frac{1}{2\hat b^2}(1+\beta a_\circ)^{\theta}})(-\frac{1}{\hat b^3}(1+\beta x)^{\theta} e^{-\frac{1}{2\hat b^2}(1+\beta x)^{\theta}})}{(1-e^{-\frac{1}{2\hat b^2}(1+\beta a_\circ)^{\alpha}})^2}$

$$-\frac{(1-e^{-\frac{1}{2\hat{b}^2}(1+\beta x)^{\theta}})(-\frac{1}{\hat{b}^3}(1+\beta a_{\circ})^{\theta}e^{-\frac{1}{2\hat{b}^2}(1+\beta a_{\circ})^{\theta}})}{(1-e^{-\frac{1}{2\hat{b}^2}(1+\beta a_{\circ})^{\alpha}})^2}] = 0$$

Then,

$$\sum_{i=1}^{n}[(\frac{1-e^{-\frac{1}{2\hat{b}^2}(1+\beta x)^{\theta}}}{1-e^{-\frac{1}{2\hat{b}^2}(1+\beta a_{\circ})^{\theta}}})-\frac{i}{n+1}]\times[\frac{(1-e^{-\frac{1}{2\hat{b}^2}(1+\beta a_{\circ})^{\theta}})(-\frac{1}{\hat{b}^3}(1+\beta x)^{\theta}e^{-\frac{1}{2\hat{b}^2}(1+\beta x)^{\theta}})}{(1-e^{-\frac{1}{2\hat{b}^2}(1+\beta a_{\circ})^{\alpha}})^2}$$
$$-\frac{(1-e^{-\frac{1}{2\hat{b}^2}(1+\beta x)^{\theta}})(-\frac{1}{\hat{b}^3}(1+\beta a_{\circ})^{\theta}e^{-\frac{1}{2\hat{b}^2}(1+\beta a_{\circ})^{\theta}})}{(1-e^{-\frac{1}{2\hat{b}^2}(1+\beta a_{\circ})^{\alpha}})^2}] = 0 \quad (49)$$

The $\hat{\beta}_{LES}$ could exist by deriving equation (48) concerning β.

$$\sum_{i=1}^{n}2[\left(\frac{1-e^{-\frac{1}{2b^2}(1+\hat{\beta}x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}}}\right)-\frac{i}{n+1}]\times[\frac{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})(\frac{\theta x}{2b^2}(1+\hat{\beta}x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\hat{\beta}x)^{\theta}})}{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})^2}$$
$$-\frac{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}x)^{\theta}})(\frac{\theta a_{\circ}}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta-1}e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})}{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})^2}] = 0$$

Then

$$\sum_{i=1}^{n}2[\left(\frac{1-e^{-\frac{1}{2b^2}(1+\hat{\beta}x)^{\theta}}}{1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}}}\right)-\frac{i}{n+1}]\times[\frac{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})(\frac{\theta x}{2b^2}(1+\hat{\beta}x)^{\theta-1}e^{-\frac{1}{2b^2}(1+\hat{\beta}x)^{\theta}})}{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})^2}$$
$$-\frac{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}x)^{\theta}})(\frac{\theta a_{\circ}}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta-1}e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})}{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})^2}] = 0 \quad (50)$$

The $\hat{\theta}_{LES}$ can be found by deriving the equation (48) with respect to θ.

$$\sum_{i=1}^{n}2[\left(\frac{1-e^{-\frac{1}{2b^2}(1+\beta x)^{\hat{\theta}}}}{1-e^{-\frac{1}{2b^2}(1+\beta a_{\circ})^{\hat{\theta}}}}\right)-\frac{i}{n+1}]\times[\frac{(1-e^{-\frac{1}{2b^2}(1+\beta a_{\circ})^{\hat{\theta}}})(\frac{1}{2b^2}(1+\beta x)^{\hat{\theta}}ln(1+\beta x)e^{-\frac{1}{2b^2}(1+\beta x)^{\hat{\theta}}})}{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})^2}$$
$$-\frac{(1-e^{-\frac{1}{2b^2}(1+\beta x)^{\hat{\theta}}})(\frac{1}{2b^2}(1+\beta a_{\circ})^{\hat{\theta}}\ln(1+\beta a_{\circ})e^{-\frac{1}{2\hat{b}^2}(1+\beta a_{\circ})^{\hat{\theta}}})}{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})^2}] = 0$$

Then,

$$\sum_{i=1}^{n}[\left(\frac{1-e^{-\frac{1}{2b^2}(1+\beta x)^{\hat{\theta}}}}{1-e^{-\frac{1}{2b^2}(1+\beta a)^{\hat{\theta}}}}\right)-\frac{i}{n+1}]\times[\frac{(1-e^{-\frac{1}{2b^2}(1+\beta a_{\circ})^{\hat{\theta}}})(\frac{1}{2b^2}(1+\beta x)^{\hat{\theta}}ln(1+\beta x)e^{-\frac{1}{2b^2}(1+\beta x)^{\hat{\theta}}})}{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})^2}$$
$$-\frac{(1-e^{-\frac{1}{2b^2}(1+\beta x)^{\hat{\theta}}})(\frac{1}{2b^2}(1+\beta a_{\circ})^{\hat{\theta}}\ln(1+\beta a_{\circ})e^{-\frac{1}{2\hat{b}^2}(1+\beta a_{\circ})^{\hat{\theta}}})}{(1-e^{-\frac{1}{2b^2}(1+\hat{\beta}a_{\circ})^{\theta}})^2}] = 0 \quad (51)$$

Equations (49) through (51) have numerical solutions.

11. Application

Data set 1**:** have data set represents the lifetime of 51 devices [23]
the data set is
0.1,0.2,1,1,1,1,1,2,3,6,7,1,1,12,18,18,18,18,18,21,32,36,40,45,46,47,50,55,
60,63,63,67,67,67,67,72,75,75,79,82,82,84,84,84,85,85,85,85,85,86,86.
Then compares the proposed method to the truncated Rayleigh Lomax distribution, the RLD, the LD, and the RPD, and it applies the MLE method to the dataset (TRPD). These metrics (CAIC, AIC, BIC, and HQIC) are used to make the model selection.

$$AIC = -2\hat{\ell} + 2q\,, \quad (52)$$

$$BIC = -2\hat{\ell} + q\log(n), \quad (53)$$

$$HQIC = -2\hat{\ell} + 2q\log(\log(n)) \quad (54)$$

$$CAIC = -2\hat{\ell} + \frac{2qn}{n-q-1} \quad (55)$$

Table (1) displays the model-level estimates (MLEs) for the data models, while Table (2) provides numerical amount for the model-chosen statistical (AIC, $\hat{\ell}$, CAIC, BIC, and HQIC). Based on Table (2), the (TRLD) model provides the smallest representation of the data set due to its small amount for the criteria CAIC, BIC, AIC, and HQIC.

Table 1. Data-based estimates of various parameters.

Models	Parameters Estimation		
TRPD(x;α,q,w)	$\hat{\alpha}$=0.66	$\hat{q}$=20.247	$\hat{w}$=10.959
R-LD(x;β,θ,b)	$\hat{\beta}$=0.844	$\hat{\theta}$=12.7726	$\hat{b}$=9.738
LD(x;α,λ)	$\hat{\alpha}$=1.08	$\hat{\lambda}$=20.744	
R-PD(x;p,b,α)	$\hat{\alpha}$=0.172	$\hat{b}$=0.627	$\hat{p}$=0.419
TRLD(x;β,bθ)	$\hat{\beta} = 0.859$	$\hat{\theta} = 2.987$	$\hat{b} = 5.617$

Table 2. $\hat{\ell}$, CAIC, BIC, AIC, and HQIC Data Set statistics values.

Models	$\hat{\ell}$	AIC	BIC	HQIC	CAIC
TRPD(x;α,b,p)	-217.7661	441.5321	447.2682	443.7164	442.0588
R-LD(x;β,θ,b)	-238.5086	495.0171	504.7532	497.2494	495.5388
LD(x;α,λ)	-248.2744	502.5487	508.2848	504.7330	503.0704
R-PD(x;p,b,α)	-287.7194	581.4387	587.1748	583.6230	581.9604
TRLD(x;β,θ,b)	-239.2239	484.4478	490.1839	486.6321	484.9695

By comparing the results in Tables (1) and (2), we conclude that the truncated Rayleigh-Pareto distribution is preferable to Rayleigh-Lomax distribution Lomax distributions, truncated Rayleigh-Lomax distributions, and Rayleigh-Pareto distributions. This is because the (TRPD) and (TRLD) distributions have the smallest values for the (BIC), (AIC), (HQIC), and (CAIC) compared to the other distributions.

12. CONCLUSIONS

The current study presents a truncated Rayleigh - Pareto distribution (TRPD) for three parameters. In addition to the stress strength reliability pdf and cdf, distribution pdf and cdf are also located. Mod, the moment generating function, the median, the quintile, the mean and variance, the moments, the geometric mean, the harmonic mean, and the order statistics were among the data we retrieved. Essential estimating techniques were also used. It is observed that the (TRPD) was preferable to other distributions we considered.

13. FUTURE WORK

The current research suggests the following future works:

1. Split the amputated period into (n) of the periods.
2. Comparison of estimation methods.
3. Use other new methods to estimate parameters.
4. Find the parameters for the (TRPD) by numerical methods
5. Find new distributions by truction.

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