Weakly Quotient Map and Space

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Abstract- **In this paper, we introduce the new weakly quotient map (briefly, *w*-quotient map), strongly *w*-quotient map and *w**-quotient map. Here will discuss composition of few such quotient maps. Also, we investigate some important properties and consequences associated with usual quotient and bi, tri-quotient maps.**

Key Words: w-quotient map, strongly w-quotient map, w*-quotient map, w-quotient space.
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I. INTRODUCTION

Construction is needed in any branch of science, which can be sufficiently achieved by the theory of quotient notion in Mathematics. We can see how identifications explain or yield simple objects like circle, sphere, suspension, and torus. In fact not only special objects, most of the objects are actually quotient creations. Allen Hetcher [1] has discussed mapping cylinder, mapping cone etc. This takes frontiers part of abstract quotient construction. Mobius strips, Klein bottle are quotient spaces, which have classical importance having special topological properties. Visualization is new aspect in the theory of gluing, but many time we unable to catch the object.

There are many situations in topology where we build a topological space by starting with some set (space) and doing some kind of "gluing" or "identifications" that makes suitable object for our claim. In the literature of Mathematics, quotient maps are generally called strong continuous maps or identification maps, because of strong conditions of continuity i.e. $h^{-1}(V)$ is open in M if and only if V is open in N of a surjective fiunction $h: M \to N$ and their importance for the philosophy of gluing. [9] In 2000, M.Sheik John, introduced and studied w-closed sets respective properties map in topological space. Lellis Thivagar [7], Chidanand Badiger et al. [2] and Balamani et al. [10] discussed generalized quotient maps, rw-quotient map and $\psi^*\alpha$-quotient maps in topological space respectively. Bi-quotient, Tri-quotient maps studied by E. Michal [4,3].

We introduce such class of maps called *w*-quotient map, strongly *w*-quotient map, *w**-quotient map in topological space. Which have many rich consequences concern to usual quotient and bi, tri-quotient maps. Here we have given example and counter examples for respective results.

II. PRELIMINARIES

Throughout this paper L, M, N, R and S represent the topological spaces on which no separation axioms are assumed unless otherwise mentioned. Map here mean function and for a subset F of topological space M, the $M \backslash F$ denotes the complement of F in M. We recall the following definitions.

Definition 2.1 [9] A subset F of a space M is said to be Weakly closed (briefly, w-closed) if $cl(F) \subseteq U$ whenever $F \subseteq U$ and U is semi-open in M. Respectively called w-open set, if $M \setminus F$ is w-closed set in M. We denote the set of all w-open sets in M by $wO(M)$.

Definition 2.2 A map $h: M \to N$ is said to be *w*-continuous map, if $h^{-1}(F)$ is *w*-closed set of M, for every closed set F of N.

Definition 2.3 A map $h: M \to N$ is said to be *w*-irresolute map, if $h^{-1}(F)$ is *w*-closed set of M, for every *w*-closed set F of N.

Definition 2.4 A bijection $h: M \to N$ is called *w*-homeomorphism, if both h and h^{-1} are *w*-continuous.

Definition 2.5 (Consider) A topological space M is called η_w-space, if every *w*-closed set is closed set.

Theorem 2.6 Every *w*-irresolute maps are *w*-continuous.

III. *w*-QUOTIENT MAP

We introduce here some class of maps *w*-quotient map, strongly *w*-quotient map, *w**-quotient map in topological space. Investigate some relations between them with usual quotient maps. Also we characterize few notions.

Definition 3.1 Let M and N be two topological spaces then a surjective map $h: M \to N$ is said to be *w*-quotient map if h is *w*-continuous and $h^{-1}(V)$ is open in M implies V is *w*-open set in N.

Example 3.2 Let $h: (\mathbb{R}, \eta_u) \to (\mathbb{R}, \eta_u)$ and η_u be usual topology on $\mathbb{R}$, the map h is defined by $h(x) = 5x$ is obviously (See 3.3) *w*-quotient map.

Theorem 3.3 Every quotient map is *w*-quotient map.

Proof: Let $h: M \to N$ be any quotient map, by definition h is surjective. It is known that every continuous map is *w*-continuous map, hence h is *w*-continuous. Since h is a quotient map, $h^{-1}(V)$ is open in M, implies V is open in N. As every open set is *w*-open set, implies V is *w*-open set in N. Therefore h is *w*-quotient map.

Theorem 3.4 [5; § 22] If $h: M \to N$ is *w*-continuous and $k: N \to M$ is continuous, such that $h \circ k: N \to N$ is identity then h is *w*-quotient map.

Proof: Since $h \circ k = Id$ implies h is bijective. h is *w*-continuous by hypothesis. For any $V \subset N$ with $h^{-1}(V)$ be open in M, continuity of k gives the $k^{-1}(h^{-1}(V)) = (h \circ k)^{-1}(V) = Id^{-1}(V) = V$ is open in N, implies V is *w*-open set in N. Therefore h is *w*-quotient map.

Theorem 3.5 h become *w*-quotient map, whenever $h: M \to N$ is surjective, continuous and open map.

Proof: By theorem 3.3, h is *w*-quotient map, because every $h: M \to N$ is surjective, continuous and open maps are quotient map.

Theorem 3.6 h become *w*-quotient map, whenever $h: M \to N$ is surjective, continuous and closed map.

Theorem 3.7 h become w-quotient map, whenever $h: M \to N$ is surjective, w-continuous and w-open map.

Theorem 3.8 h become w-quotient map, whenever $h: M \to N$ is surjective, w-continuous and w-closed map.

Proof: Two conditions are obvious by the hypothesis. For last condition, any $V \subset N$ with $h^{-1}(V)$ be open in M, implies $M \setminus h^{-1}(V)$ is closed set in M. Since h is w-closed map implies $h(M \setminus h^{-1}(V)) = N \setminus V$ is w-closed in N. Therefore V is w-open set in N, h is w-quotient map.

IV. STRONGLY w-QUOTIENT AND w^*-QUOTIENT MAP

Definition 4.1 Let M and N be two topological spaces, a surjective map $h: M \to N$ is said to be strongly w-quotient map provided $V \subset N$ is open in N if and only if $h^{-1}(V)$ is w-open in M.

Definition 4.2 Let M and N be two topological spaces, a surjective map $h: M \to N$ is said to be w*-quotient map if h is w-irresolute and $h^{-1}(V)$ is w-open in M implies V is open set in N.

Theorem 4.3 Every injective w*-quotient map is w*-open map.

Proof: Let $h: M \to N$ be any injective w*-quotient map, For every w-open set V in M, injective gives $h^{-1}(h(V)) = V$ is w-open set in M. Since h is w*-quotient map, implies $h(V)$ is open set as well w-open in N.

Theorem 4.4 Every injective w*-quotient map is w*-closed map.

Proof: Let $h: M \to N$ be injective w*-quotient map, For every w-closed set F in M, implies $M \setminus F$ is w-open set in M. Obvious $h(M \setminus F) = N \setminus h(F)$ is w-open set in N. Hence $h(F)$ is w-closed set in N.

Theorem 4.5 Every strongly w-quotient map is w-quotient map.

Proof: Let $h: M \to N$ be strongly w-quotient map, obviously the first two conditions hold. For $V \subset N$ with $h^{-1}(V)$ be an open set in M, also that become w-open in M. Since h is strongly w-quotient map, implies V is open set in N, therefore V is w-open set in N.

Theorem 4.6 Every w*-quotient map is strongly w-quotient map.

Proof: Let $h: M \to N$ be strongly w*-quotient map, two conditions are obvious. Because V is any open set in N is also w-open set in N. Since h is w-irresolute implies $h^{-1}(V)$ is w-open set in M. For $W \subset N$ with $h^{-1}(W)$ be an open set in M, implies $h^{-1}(W)$ is w-open in M. Since h is w*-quotient map, implies W is open set in N.

Theorem 4.7 Every w*-quotient map is w-quotient map.

Proof: It followed by theorem 4.5 and 4.6

V. CONSEQUENCES ON COMPOSITIONS

Theorem 5.1 Composition of two quotient maps is w-quotient map.

Proof: We know composition of two quotient maps is quotient map [5; 3.29]. Hence by theorem 3.3 which is w-quotient map.

Remark 5.2 Composition of quotient map with w-quotient map need not be w-quotient map.

Remark 5.3 Composition of w-quotient map with quotient map need not be w-quotient map.

Remark 5.4 Composition of two w-quotient maps need not be w-quotient map.

Theorem 5.5 If N is η_w topological space, $h: M \to N$ is w-quotient map and $k: N \to R$ is quotient map then $k \circ h$ is w-quotient map.

Proof: Obviously $k \circ h$ is surjective and for every open set $U \subset R$, the $(k \circ h)^{-1}(U) = h^{-1}(k^{-1}(U))$ which is w-open set in M, implies $k \circ h$ is w-continuous. For every $U \subset R$ with $(k \circ h)^{-1}(U) = h^{-1}(k^{-1}(U))$ is open in M, implies $k^{-1}(U)$ is w-open set in N, by the hypothesis k is quotient map. Hence U is open set as well w-open set in R.

Theorem 5.6 If N is η_w topological space, $h: M \to N$ is quotient map and $k: N \to R$ is w-quotient map then $k \circ h$ is w-quotient map.

Proof: Similar arguments with essential changes in 5.5 can works.

Corollary 5.7 If N is η_w topological space, $h: M \to N$ is w-quotient map and $k: N \to R$ is w-quotient map then $k \circ h$ is w-quotient map.

Proof: It fallows by combining the theorems 5.5 and 5.6 and (Converse need not hold).

Theorem 5.8 If $h: M \to N$ is a strong w–quotient map and $k: N \to R$ is a quotient map then koh is strong w-quotient map.

Proof: Here surjective of k and h gives koh is surjective and koh is w–continuous map due to composition of continuous and w- continuous. Lastly if $h^{-1}(k^{-1}(V))$ is open in M. Since h is strong w-quotient map and quotient of k implies V is open in R. Therefore koh is strong w-quotient map.

Theorem 5.9 If $h: M \to N$ is open surjective, w-irresolute and $k: N \to R$ is w-quotient map then $k \circ h$ is w-quotient map.

Proof: Obviously $k \circ h$ surjective. For every open set $U \subset R$, the k is w-continuous and h is w-irresolute gives $(k \circ h)^{-1}(U) = h^{-1}(k^{-1}(U))$ is w-open set in M. Now for every $W \subset R$ with $(k \circ h)^{-1}(W) = h^{-1}(k^{-1}(W))$ be open in M. Since h is open map $h\left(h^{-1}(k^{-1}(W))\right) = k^{-1}(W)$ is open in N, given k is w-quotient map implies W is w-open set in R. Hence $k \circ h$ is w-quotient map.

Theorem 5.10 If $h: M \to N$ is w*-open surjective and w-irresolute and $k: N \to R$ is strongly w-quotient map then $k \circ h$ is strongly w-quotient map.

Proof: Obviously $k \circ h$ surjective and w-continuous. For every $U \subset R$ with $(k \circ h)^{-1}(U) = h^{-1}(k^{-1}(U))$ be w-open in M, since h is w*-open map implies $h(h^{-1}(k^{-1}(U))) = k^{-1}(U)$ is w-open in N. Since k is strongly w-quotient map implies U is open set in R. Hence $k \circ h$ is strongly w-quotient map.

Theorem 5.11 If $h: M \to N$ is w*-open, surjective and w-irresolute and $k: N \to R$ is w*-quotient map then $k \circ h$ is w*-quotient map.

Proof: Arguments in 5.9 and 5.10 yield the proof.

Theorem 5.12 If $h: M \to N$ and $k: N \to R$ are w*-quotient maps then $k \circ h$ is w*-quotient map.

Proof: Obviously $k \circ h$ surjective and w irresolute. For every $U \subset R$ with $(k \circ h)^{-1}(U) = h^{-1}(k^{-1}(U))$ be w-open in M, since h is w*-quotient map implies $k^{-1}(U)$ is open and w-open in N. Since k is w*-quotient map implies U is open set in R. Hence $k \circ h$ is w*-quotient map.

VI. STANDARD COMPARISONS AND APPLICATIONS

Theorem 6.1 If h: M → N is any map, where M and N are η_w topological spaces then following are equivalent.

i) h is *w**-quotient map
ii) h is strongly *w*-quotient map
iii) h is *w*-quotient map

Proof: (i) ⇒ (ii) Surjective is obvious and since every *w*-irresolute map is *w*-continuous map. Third condition is trivial by the definition of *w**-quotient map.
(ii) ⇒ (iii) Obvious by the similar arguments in above.
(iii) ⇒ (i) Surjective is obvious and h is *w*-irresolute, because every continuous map is *w*-irresolute because N is η_w. Let $h^{-1}(V)$ be *w*-open set in M, implies $h^{-1}(V)$ is open set in M. The *w*-quotient map of h implies V is *w*-open set and open set in N.

Theorem 6.2 If $h:(M,\eta) \to (N,\xi)$ is surjective, *w*-continuous map then following are equivalent.

i) h is strongly *w*-quotient map.
ii) For any $k:(N,\xi) \to (R,\tau)$ then k is continous if and only if $k \circ h$ is *w*-continuous.
iii) For fixed topology η on M then ξ is the maximal topology for h to be *w*-continuous.

Proof: (i) ⇒ (ii) For every open set $U \subset R$, the $(k \circ h)^{-1}(U) = h^{-1}(k^{-1}(U))$ is *w*-open set in (M,η) because of k is continuous and h is *w*-continuous. Therefore $k \circ h$ is *w* continuous. Conversely, for every open set U of (R,τ), the $(k \circ h)^{-1}(U) = h^{-1}(k^{-1}(U))$ is *w*-open set in (M,η) since h is strongly *w*-quotient map gives $k^{-1}(U)$ is open in (N,ξ). Therefore k is continuous.
(ii) ⇒ (iii) On contrary, with fixed toplogy η on M, there exists another topology $\xi' \supsetneq \xi$ such that $h:(M,\eta) \to (N,\xi')$ is *w*-continuous. Obviously we see identity map $id:(N,\xi) \to (N,\xi')$ become not continuous, because $\xi' \supsetneq \xi$. But $id \circ h = h:(M,\eta) \to (N,\xi')$ become *w*-continuous. Hence $id \circ h$ is *w*-continuous which contradicts to the hypothesis in (ii). Therefore ξ is the maximal topology for h is *w*-continuous.
(iii) ⇒ (i) Surjective and *w*-continuous of h are obvious. For last condition, on contrary, there exist a $V_0 \subset N$ with $h^{-1}(V_0)$ is *w*-open set in (M,η), but V_0 is not an open set in (N,ξ). Let $\mathcal{B} = \xi \cup \{V_0\}$, which induces topology ξ^* on N, which contains V_0, also $\xi^* \supsetneq \xi$. But $h:(M,\eta) \to (N,\xi^*)$ also *w*-continuous. Which contradicts to (iii), therefore h is strongly *w*- quotient map.

Theorem 6.3 [5; § 22] If $h: M \to N$ is *w*-continuous and $k: N \to M$ is continuous, such that $h \circ k: N \to N$ is identity with N is η_w- topological space then h is *w**-quotient map.
Proof: Arguments in theorem 3.6 and N is η_w-space can give the proof.

Theorem 6.4 [5; §22] If $h: M \to N$ is a strong *w*–quotient map and $k: M \to R$ is a map that is constant on each set $h^{-1}(\{y\})$ for $y \in N$, then
i) k induces a map $l: N \to R$ such that $loh = k$.
ii) The induced map l is continuous iff k is *w*-continuous.
iii) The induced map l is quotient iff k strong *w*-quotient map.
Proof: i) Since k is constant on $h^{-1}(\{y\})$ for $y \in N$, the set $k(h^{-1}(\{y\}))$ is a one point set in R. By considering l(y) denote this point, then which is clear that l is well–defined on N and can see as each $x \in M$, $l(h(x)) = k(x)$.
ii) If l is continuous and h is *w*-continuous implies $loh = k$ is *w* continuous. On other hand let U be any open set in R then $k^{-1}(U)$ is an *w*-open set due to k is *w*-continuous. But $k^{-1}(U) = h^{-1}(l^{-1}(U))$ is *w*-open in M. Since h is a strong *w*–quotient map, $l^{-1}(U)$ is a open set. Hence l is continuous.
iii) If l is quotient map and h is *w*–quotient map by theorem 5.11, $loh = k$ is strong *w*-quotient map. On other hand, since $loh = k$ surjective implies l is surjective and l is continuous by above result (ii). If $l^{-1}(U)$ is open in N and *w*-continuity of h implies $h^{-1}(l^{-1}(U)) = k^{-1}(U)$ is *w*-open in M. Since k is strong *w*-quotient map implies U is open in R. Hence l is quotient map.

The following theorems 6.5 to 6.13 followed obviously from the results of [4] i.e. every bi-quotient map is quotient map and from [4,9] implies that every tri quotient maps are quotient maps Hence by theorem 3.3 both class of maps become h as *w*-quotient maps.

Theorem 6.5 Arbitrary product map of bi quotient maps are *w*-quotient.

Theorem 6.6 If N is Hausdorff space and any map $h: M \to N$ is surjective continuous then h and $h \times Id_R$ are *w*-quotient maps for every space R.

Theorem 6.7 If N is Regular space , map $h: M \to N$ is surjective continuous map and $h \times k$ is quotient maps for every quotient maps k then h *w*-quotient maps.

Theorem 6.8 If maps $h: M \to N$ and $k: R \to S$ are quotient maps and M and $N \times S$ are Hausdorff k-spaces then $h \times k$ is *w*-quotient map.

Theorem 6.9 Any surjective contiguous map h from M to a hausdorff space N is *w*-quotient map whenever h is compact covering, and N is locally compact.

Theorem 6.10 The $h: M \to N$ is continous surjective map become *w*-quotient map whenever either h is perfect or h is compact-covering and Y is locally compact and Hausdorff.

Theorem 6.11 If M regular sieve-complete space, N is paracompact space then every inductively perfect map $h: M \to N$ is *w*-quotient.

Theorem 6.12 If M completely metrizable, N is paracompact space then following gives $h: M \to N$ is *w*-quotient.
(i) Either N is metrizable or N is completely metrizable.
(ii) Y is a countably bi-k-space.

Theorem 6.13 Product map $h \times k: M \times R \to N \times S$ is *w*-quotient map whenever $h: M \to N$ and $K: R \to S$ are tri-quotient.

Theorem 6.14 Suppose $h: M \to N$ and $k: N \to R$ are continuous maps then
(i) If h and k are tri-quotient, or bi quotient, or quotient then is koh is *w*-quotient map.
(ii) If koh is tri-quotient, or bi quotient, or quotient, or *w* quotient, or strong *w* quotient map then is k is *w*-quotient map.
Proof: i) Composition of two tri-quotients, or bi quotients, or quotients are respectively tri-quotients, or bi quotients, or quotients. Hence by Theorem 3.3 koh is *w*-quotient map.
ii) Refer [3;Theorem 7.1 and Theorem 3.6] or Arguments are easy.

Theorem 6.15 A map $h: M \to N$ is harmonious then h is w-quotient map.
Proof: Arguments in [8; Proposition 3.8] says harmonious implies tri quotient and [3,8] gives the proof.

Construction 6.16 Let M be a topological space and N be a set. Let $h: M \to N$ be any surjective map construct a set $Q\eta_w = \{ V \subset N \mid h^{-1}(V) \text{ is } w - \text{open set in } M \}$ then obviously the collection $Q\eta_w$ is not nessasarly topology on N (See basic union intersection properties of two w-open sets need obey axioms of topology) but some time (If M with topology η_w then clear that $Q\eta_w$ become topology on N) this become topology. With this topology on N, $h: M \to N$ become w-quotient map and quotient map. This topology $Q\eta_w$ we are calling w-quotient topology and $(N, Q\eta_w)$ is called w-quotient topological space.

With which we can see weak sense of gluing or identifying or joining in standard constructions like Cone, cylinder, Sphere, Mobius strip, Torus, Klein bottle and suspension etc. [1].

For standard example If $\sim$ is any equivalence relations on M with η_w then cannanical projection $\rho: M \to M/\!\sim$ by, $\rho(x) = [x]$ is clear w-quotient map, $M/\!\sim$ is w-quotient topological space under w-quotient topology $Q\eta_w$.

Theorem 6.17 If M with topology η_w , $h: M \to N$ is w quotient map and $M^* = \{h^{-1}(\{y\}) : y \in N\}$ then N is w-homeomorphic to M^*.
Proof. M^* is $Q\eta_w$ topological space From theorem 3.58. define the map $k: M^* \to N$, by $k([x]) = h(x)$ became w-homeomorphism.

Application 6.18 The w-quotient map and w-quotient space are finding applications in gluing in some weak sense. Discussed map and space are weak sense of gluing or identifications and weakly glued space. As seen every gluing is w-gluing but not converse. Anyone can see the characterization notion that, If M is η_w given any space N and map from M to N then “the map is w-quotient map if and only if it is quotient map”. Hence gluing and weak gluing are same when M (involved space) is a η_w space.

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REFERENCES

[1] A. Hetcher, *Algebraic Topology*, Cambridge University press (2002).
[2] Chidanand Badiger, T.Venkatesh and Basayya B. Mathad, *Regular weakly quotient map and space*, Malaya Journal of Matematik, Vol. S, No. 1, 115-120, 2020.
[3] E. Michel, *Complete Spaces And Tri-Quotient Maps*, University Of Washington Seattle, Washington, (1976), 716-733.
[4] E. Michel, *Bi-quotient maps and Cartesian products of quotient maps*, Annales de l’ institute Fourier, tome, 18,2 (1968), 287-302.
[5] J. R. Munkres, *Topology*, Prentice Hall India Learn. Priv. Ltd; Second edition, (2002).
[6] J. M. Lee, *Introduction to Topological Manifolds*, springer, (2000).
[7] L. Thivagar, *A Note on quotient mappings*, Bull, Malaysian math.Sci.(Second series,, 14(1991), 21-30.
[8] M. Pillot, *Tri-quotient maps become inductively perfect with the aid of consonance and continuous selections*, Topology and its Applications, 104(2000), 237-253.
[9] M.Sheik John, *On w-closed sets in topology*, ActaCienciaIndica,, 4(2000), 389-392.
[10] N. Balamani and A.Parvati, *On topological $\psi^*\alpha$ -quotient mappings*, Int. J. of Math. Trends and Tech, 48(3)(2017), 209-213.